Notes on holographic Sasaki-Einstein backgrounds

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Why Sasaki-Einstein and holography

• The AdS/CFT: relates a SUGRA in the $AdS_5 \times X_5$ to a strongly coupled, rank N, SCFT on the 4-d flat boundary $R^{3,1}$ of AdS_5.

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• Holographic dual of Sasaki-Einstein: quiver theories

• Recently: metric g_M on M emerges from the canonical ensemble (of N "point particles")in the large N-limit $\Longrightarrow$ emergent Sasaki-Einstein

Sasaki-Einstein $Y^{p,q}$

The metric tensor of $Y^{p,q}$ parameterized by two positive integers p, q $(p > q)$

$$
\begin{aligned}
ds^2 = \frac{1-y}{6}\left(d\theta^2+\sin^2\theta d\phi^2\right)+\frac{1}{w(y)q(y)}dy^2+\frac{q(y)}{9}(d\psi-\cos\theta d\phi)^2 \\
+w(y)\left[d\alpha+f(y)(d\psi-\cos\theta d\phi)\right]^2 \equiv ds^2(B)+w(y)(d\alpha+A)^2.
\end{aligned}
$$

The functions are

$$
\begin{aligned}
w(y) &= \frac{2(b-y^2)}{1-y}\ ,\ q(y)=\frac{b-3y^2+2y^3}{b-y^2}\ ,\ f(y)=\frac{b-2y+y^2}{6(b-y^2)}\ , \\
b &= \frac{1}{2}-\frac{p^2-3q^2}{4p^3}\sqrt{4p^2-3q^2}\ . \qquad (1)
\end{aligned}
$$

The coordinates $\{y, \theta, \phi, \psi, \alpha\}$ have the following ranges $(0 < b < 1)$:

$$
y_1 \le y \le y_2\ ,\ 0 \le \theta \le \pi\ ,\ 0 \le \phi \le 2\pi\ ,\ 0 \le \psi \le 2\pi\ ,\ 0 \le \alpha \le 2\pi l\ . \qquad (2)
$$

Details of the $Y^{p,q}$ geometry

- Base manifold

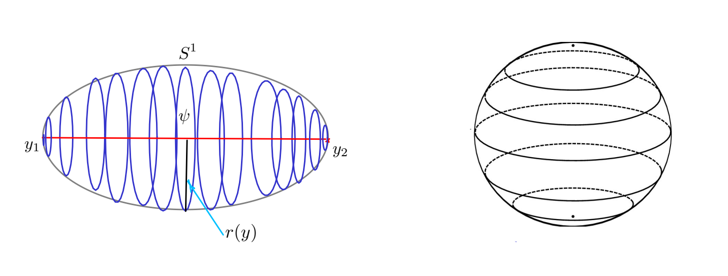

Figure: Squashed sphere as circle fibration parametrized by ψ over the interval $[y_1, y_2]$ and round sphere.

The topology of the base is $B \cong S^2 \times S^2$

- S^1 principle bundle over the base $\quad \rightarrow \quad d\alpha + A$
- Isometries are $SU(2) \times U(1) \times U(1)$.

Schrödinger equation $\Box\,\Phi = -E\,\Phi$ with

$$\Box = \frac{1}{1-y}\frac{\partial}{\partial y}(1-y)w(y)q(y)\frac{\partial}{\partial y}$$
$$+\left(\frac{3}{2}\hat{Q}_R\right)^2 + \frac{1}{w(y)q(y)}\left(\frac{\partial}{\partial\alpha} + 3y\hat{Q}_R\right)^2 + \frac{6}{1-y}\left[\hat{K} - \left(\frac{\partial}{\partial\psi}\right)^2\right]. \quad (3)$$

The R-symmetry operator is $\hat{Q}_R = 2\partial_\psi - 1/3\partial_\alpha$ and $\hat{K}$ is the second Casimir of $SU(2)$ - a part of the isometry $SU(2)\times U(1)^2$,

$$\hat{K} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi} + \cos\theta\frac{\partial}{\partial\psi}\right)^2 + \left(\frac{\partial}{\partial\psi}\right)^2 \quad (4)$$

Due to the isometry, the eigenfunction takes the form

$$\Phi(y,\theta,\phi,\psi,\alpha) \;=\; \exp\left[i\left(P_\phi\phi + P_\psi\psi + \frac{P_\alpha}{l}\alpha\right)\right]Y(y)\Theta(\theta) \quad (5)$$

with $P_\phi, P_\psi, P_\alpha \in \mathbb{Z}$, $\hat{K}$ acting on $SU(2)$ part.

The regular solutions of the equation below are given by Jacobi polynomials.

$$\hat{K}\underbrace{e^{i\left(P_\phi\phi+P_\psi\psi\right)}\Theta(\theta)}_{SU(2)\text{ part}} = -J(J+1)\,e^{i\left(P_\phi\phi+P_\psi\psi\right)}\Theta(\theta)\ , \tag{6}$$

The rest

$$\frac{1}{1-y}\frac{d}{dy}\left[(1-y)w(y)q(y)\frac{d}{dy}Y(y)\right] - \left[\left(\frac{3}{2}Q_R\right)^2 + \frac{1}{w(y)q(y)}\left(\frac{P_\alpha}{l}+3yQ_R\right)^2 + \frac{6}{1-y}\left(J(J+1)-P_\psi^2\right) - E\right]Y(y) = 0\ .$$

converts into Fuchsian-type with four regular singularities at $y = y_1, y_2, y_3$ and ∞, i.e. Heun's equation;

$$\frac{d^2}{dy^2}Y(y) + \left(\sum_{i=1}^{3}\frac{1}{y-y_i}\right)\frac{d}{dy}Y(y) + o(y)Y(y) = 0\ , \tag{7}$$

The functions and parameters

$$
\begin{aligned}
o(y) &= \frac{1}{P(y)}\left[\mu - \frac{y}{4}E - \sum_{i=1}^{3}\frac{\alpha_i^2 P'(y_i)}{y-y_i}\right]\ ,\ P(y) = \prod_{i=1}^{3}(y-y_i)\ , \\
\mu &= \frac{E}{4} - \frac{3}{2}J(J+1) + \frac{3}{2}\left(\frac{2}{3}\frac{P_\alpha}{l} - Q_R\right)^2 \qquad (8)
\end{aligned}
$$

where $l = \frac{q}{3q^2-2p^2+p\sqrt{4p^2-3q^2}}$ and

$$
\begin{aligned}
\alpha_1 &= \pm\frac{1}{4}\left[P_\alpha\left(p+q-\frac{1}{3l}\right) - Q_R\right]\ , \qquad (9) \\
\alpha_2 &= \pm\frac{1}{4}\left[P_\alpha\left(p-q+\frac{1}{3l}\right) + Q_R\right]\ , \qquad (10) \\
\alpha_3 &= \pm\frac{1}{4}\left[P_\alpha\left(\frac{-2p^2+q^2+p\sqrt{4p^2-3q^2}}{q} - \frac{1}{3l}\right) - Q_R\right]\ . \qquad (11) \\
y_{1,2} &= \frac{1}{4p}\left(2p \mp 3q - \sqrt{4p^2-3q^2}\right)\ , y_3 = \frac{1}{2} + \frac{\sqrt{4p^2-3q^2}}{2p}\ . \qquad (12)
\end{aligned}
$$

It is convenient to transform the singularities from $\{y_1, y_2, y_3, \infty\}$ to $\{0, 1, t = \frac{y_1 - y_3}{y_1 - y_2}, \infty\}$. This is achieved by the transformation

$$\boxed{x = \frac{y - y_1}{y_2 - y_1}} \tag{13}$$

together with the rescaling

$$\boxed{Y = x^{\alpha_1}(1-x)^{\alpha_2}(t-x)^{\alpha_3} q(x)\ ,} \tag{14}$$

which transforms (7) to the standard form of Heun's equation

$$\boxed{\frac{d^2}{dx^2} q(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-t}\right) \frac{d}{dx} q(x) + \frac{\alpha\beta x - k}{x(x-1)(x-t)} q(x) = 0}$$

Bunch of Heun's parameters

$$\alpha = -\lambda + \sum_{i=1}^{3} |\alpha_i| \ , \ \beta = 2 + \lambda + \sum_{i=1}^{3} |\alpha_i| \ ,$$
$$\gamma = 1 + 2\alpha_1 \ , \ \delta = 1 + 2\alpha_2 \ , \ \epsilon = 1 + 2\alpha_3 \ , \tag{15}$$

The parameter k, the "accessory" parameter, is

$$\begin{aligned} k &= (|\alpha_1| + |\alpha_3|)(|\alpha_1| + |\alpha_3| + 1) - |\alpha_2|^2 \\ &\quad + t\left\{(|\alpha_1| + |\alpha_2|)(|\alpha_1| + |\alpha_2| + 1) - |\alpha_3|^2\right\} - \tilde{\mu} \end{aligned} \tag{16}$$

with

$$\begin{aligned} \tilde{\mu} &= -\frac{1}{y_1 - y_2}(\mu - y_1\lambda(\lambda + 2)) \\ &= \frac{p}{q}\left[\frac{2}{3}(1 - y_1)\lambda(\lambda + 2) - J(J+1) + \frac{1}{16}\left(\frac{2}{3}\frac{N_\alpha}{l} - Q_R\right)^2\right] , \end{aligned} \tag{17}$$

$$t = \frac{1}{2}\left(1 + \frac{\sqrt{4p^2 - 3q^2}}{q}\right) . \tag{18}$$

Note that the parameter t satisfies the inequality $t > 1$ reflecting $p > q$.

A little holography of point-like string in $Y^{p,q}$

- **Point-like strings** $Y^{p,q}$

$$S = \frac{\sqrt{\lambda}}{2}\int d\tau\left(-\dot{t}^2 + g_{ab}\dot{x}^a\dot{x}^b\right). \tag{19}$$

The standard equations of motion are supplemented also with the Virasoro constraint

$$-\dot{t}^2 + g_{ab}\dot{x}^a\dot{x}^b = 0. \tag{20}$$

For the metric at hand the action is reduces to

$$S = \frac{\sqrt{\lambda}}{2}\int d\tau\Big[\frac{1-y}{6}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{\omega(y)q(y)}\dot{y}^2 + \frac{q(y)}{9}(\dot{\psi}^2 - \cos\theta\dot{\phi}^2) + w(y)\left[\dot{\alpha} + f(y)(\dot{\psi} - \cos\theta\dot{\phi})\right]^2\Big]. \tag{21}$$

The Hamiltonian for the point-like string is

$$H = \frac{1}{2}g^{\mu\nu}P_\mu P_\nu. \tag{22}$$

The conjugate momenta to the coordinates $(\theta, \phi, y, \alpha, \psi)$ are:

$$
\begin{aligned}
\frac{1}{\sqrt{\lambda}} P_\theta &= \frac{1-y}{6}\dot{\theta}\,, \\
\frac{1}{\sqrt{\lambda}} P_y &= \frac{1}{6p(y)}\dot{y}\,, \\
\frac{1}{\sqrt{\lambda}} P_\alpha &= w(y)\left(\dot{\alpha} + f(y)\left(\dot{\psi} - \cos\theta\dot{\phi}\right)\right)\,, \\
\frac{1}{\sqrt{\lambda}} P_\psi &= w(y)f(y)\dot{\alpha} + \left[\frac{q(y)}{9} + w(y)f^2(y)\right]\left(\dot{\psi} - \cos\theta\dot{\phi}\right)\,, \\
\frac{1}{\sqrt{\lambda}} P_\phi &= \frac{1-y}{6}\sin^2\theta\dot{\phi} - \cos\theta P_\psi \\
&= \frac{1-y}{6}\sin^2\theta\dot{\phi} - \cos\theta w(y)f(y)\dot{\alpha} - \cos\theta\left[\frac{q(y)}{9} + w(y)f^2(y)\right]\dot{\psi} \\
&\quad + \cos^2\theta\left[\frac{q(y)}{9} + w(y)f^2(y)\right]\dot{\phi}\,,
\end{aligned}
\tag{23}
$$

where $p(y) = w(y)q(y)/6 = (b - 3y^2 + 2y^3)/[3(1-y)]$ and dot means proper time derivative.

• The momentum P_t conjugate to t is the energy of the string $\Longrightarrow$ equal to the conformal dimension Δ of the dual operator:

$$\Delta = P_t \equiv H = \sqrt{\lambda}\kappa \tag{24}$$

• The R-charge:

$$Q_R = 2P_\psi - \frac{1}{3}P_\alpha \tag{25}$$

• The energy/dispersion relations

$$\Delta^2 = \left(\frac{3}{2}Q_R\right)^2 + \frac{(P_\alpha + 3yQ_R)^2}{6p(y)} + 6p(y)P_y^2 + \frac{6(J^2 - P_\psi^2)}{1-y} \tag{26}$$

• Minimizing $H \Longrightarrow P_y = 0$; $y_0 = -\frac{P_\alpha}{3Q_R} \Longrightarrow \Delta = \frac{3}{2}Q_R \Longrightarrow$ BPS

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Summary:

a) The full set of point-like strings moving only in the transverse SE manifold is completely described by eq. (26);

b) for all BPS geodesics motion we obtain:

$$P_\alpha = -3y_0Q_R, \;\; Q_R = (2J - \frac{1}{3}P_\alpha) \;\; \Leftrightarrow \;\; \Delta = \frac{3}{2}Q_R, \;\; Q_R = 2P_\psi - \frac{1}{3}P_\alpha$$

Sasaki-Einstein $L^{p,q,r}$

- The metric

$$ds^2 = (d\tau + \sigma)^2 + \frac{\rho^2}{4\Delta_x}dx^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2$$
$$+ \frac{\Delta_x}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2 + \frac{\Delta_\theta \sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi\right)^2,$$

- Functions and parameters

$$\Delta_\theta = \alpha\cos^2\theta + \beta\sin^2\theta, \quad \sigma = \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi,$$

$$\rho^2 = \Delta_\theta - x, \quad \Delta_x = x(\alpha - x)(\beta - x) - \mu = (x - x_1)(x - x_2)(x - x_3),$$
$$\rho^2 = \Delta_\theta - x, \qquad \Delta_\theta = \alpha\cos^2\theta + \beta\sin^2\theta.$$

- In general, for a cubic equation $px^3 + qx^2 + rx + s = 0$, the roots satisfy the relations

$$x_1 + x_2 + x_3 = -q/p, \quad x_1x_2 + x_1x_3 + x_2x_3 = r/p, \quad x_1x_2x_3 = -s/p$$

- Change the variable θ by $y = \cos 2\theta \quad \Longrightarrow$

$$\sigma = \frac{(\alpha - x)(1-y)}{2\alpha} d\phi + \frac{(\beta - x)(1+y)}{2\beta} d\psi, \quad \Delta_\theta = \frac{\alpha(1+y)}{2} + \frac{\beta(1-y)}{2}$$

$$\Delta_y := (1-y^2)\left(\frac{\alpha(1+y)}{2} + \frac{\beta(1-y)}{2}\right) = \frac{\beta-\alpha}{2}(1-y^2)\left(\frac{\beta+\alpha}{\beta-\alpha} - y\right).$$

- The scalar Laplacian for the $L^{a,b,c}$ metric is given by

$$\begin{aligned}\Box_{(5)} = {} & \frac{4}{\rho^2}\frac{\partial}{\partial x}\left(\Delta_x \frac{\partial}{\partial x}\right) + \frac{4}{\rho^2}\frac{\partial}{\partial y}\left(\Delta_y \frac{\partial}{\partial y}\right) + \frac{\partial^2}{\partial \tau^2} \\ & + \frac{\alpha^2\beta^2}{\rho^2 \Delta_x}\left(\frac{(\beta - x)}{\beta}\frac{\partial}{\partial \phi} + \frac{(\alpha - x)}{\alpha}\frac{\partial}{\partial \psi} - \frac{(\alpha - x)(\beta - x)}{\alpha\beta}\frac{\partial}{\partial \tau}\right)^2 \\ & + \frac{\alpha^2\beta^2}{\rho^2 \Delta_y}\left(\frac{(1+y)}{\beta}\frac{\partial}{\partial \phi} - \frac{(1-y)}{\alpha}\frac{\partial}{\partial \psi} - \frac{(\alpha - \beta)(1-y^2)}{2\alpha\beta}\frac{\partial}{\partial \tau}\right)^2.\end{aligned} \tag{27}$$

- The $x-$singuarities are at $x_1,\ x_2,\ x_3$; the $y-$singularities are located at $y_1 = 1,\ y_2 = -1,\ y_3 = \frac{\beta+\alpha}{\beta-\alpha}$.
- Isometries: $\ell_i = -(a_i\partial_\phi + b_i\partial_\psi + c_i\partial_\tau)$, where

$$a_i = \frac{\alpha c_i}{x_i - \alpha}, \quad b_i = \frac{\beta c_i}{x_i - \beta}, \quad c_i = \frac{(\alpha - x_i)(\beta - x_i)}{2(\alpha+\beta)x_i - \alpha\beta - 3x_i^2}.$$

- Ansatz $\Psi = e^{ic_\tau \tau + ic_\phi \phi + ic_\psi \psi} X(x) Y(y) \quad \Longrightarrow$ separation of variables

The equation for separated x-system:

$$\frac{d^2}{dx^2} X(x) + \frac{\Delta_x'(x)}{\Delta_x(x)} \frac{d}{dx} X(x) + \frac{1}{4\Delta_x(x)} \Big[C - 2(\alpha c_\phi + \beta c_\psi) c_\tau + (\alpha + \beta) c_\tau^2 - Ex - \sum_i \frac{\omega_i^2 \Delta_x'(x_i)}{x - x_i} \Big] X(x) = 0.$$

The equation for separated y-system takes the form

$$\frac{d^2}{dy^2} Y(y) + \frac{\tilde{\Delta}_y'(y)}{\tilde{\Delta}_y(y)} \frac{d}{dy} Y(y) + \frac{1}{4\tilde{\Delta}_y(y)} \Big[-\frac{2C}{\beta - \alpha} + \frac{4}{\beta - \alpha} (\alpha c_\phi + \beta c_\psi) c_\tau - \frac{2(\alpha + \beta)}{\beta - \alpha} c_\tau^2 + \frac{\beta + \alpha}{\beta - \alpha} E - Ey - \sum_i \frac{v_i^2 \tilde{\Delta}_y'(y_i)}{y - y_i} \Big] Y(y) = 0,$$

where $v_1 = c_\phi, \quad v_2 = c_\psi, \quad v_3 = c_\tau - c_\phi - c_\psi.$

We found two separated Heun equations, for x- and y-systems!

A little holography of point-like string in $L^{p,q,r}$

- The metric

$$ds^2 = (d\tilde{\tau} + \sigma)^2 + \frac{\rho^2}{4\Delta_x} dx^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_x}{\rho^2}\left(\frac{\sin^2\theta}{\alpha} d\phi + \frac{\cos^2\theta}{\beta} d\psi\right)^2 + \frac{\Delta_\theta \sin^2\theta \cos^2\theta}{\rho^2}\left(\frac{\alpha - x}{\alpha} d\phi - \frac{\beta - x}{\beta} d\psi\right)^2,$$

- It is convenient to change the variable θ by $y = \cos 2\theta$. Then

$$\sigma = \frac{(\alpha - x)(1-y)}{2\alpha} d\phi + \frac{(\beta - x)(1+y)}{2\beta} d\psi, \quad \Delta_\theta = \frac{\alpha(1+y)}{2} + \frac{\beta(1-y)}{2}$$

$$\Delta_y := (1-y^2)\left(\frac{\alpha(1+y)}{2} + \frac{\beta(1-y)}{2}\right) = \frac{\beta - \alpha}{2}(1-y^2)\left(\frac{\beta+\alpha}{\beta-\alpha} - y\right).$$

The point particle action becomes

$$S = \frac{\sqrt{\lambda}}{2} \int d\tau \left[(\dot{\tilde{\tau}} + \dot{\sigma})^2 + \frac{\rho^2}{4\Delta_x} \dot{x}^2 + \frac{\rho^2}{4\Delta_y (1 - y^2)} \dot{y}^2 + \frac{\Delta_x}{\rho^2} \dot{A}^2 + \frac{\Delta_y (1 - y^2)}{4\rho^2} \dot{B}^2 \right], \quad (28)$$

where

$$\dot{\sigma} = \frac{(\alpha - x)(1 - y)}{2\alpha} \dot{\phi} + \frac{(\beta - x)(1 + y)}{2\beta} \dot{\psi},$$
$$\dot{A} = \frac{1 - y}{2\alpha} \dot{\phi} + \frac{1 + y}{2\beta} \dot{\psi}, \quad \dot{B} = \frac{\alpha - x}{\alpha} \dot{\phi} - \frac{\beta - x}{\beta} \dot{\psi}. \quad (29)$$

From the action: τ, ϕ and ψ are cyclic coordinates and we can safely set their momenta to constants

$$P_{\tilde{\tau}} = c_{\tilde{\tau}}, \qquad P_{\phi} = c_{\phi}, \qquad P_{\psi} = c_{\psi}.$$

• Momenta from the action

$$\frac{1}{\sqrt{\lambda}}P_{\tilde{\tau}} = \dot{\tilde{\tau}} + \dot{\sigma} \quad \frac{1}{\sqrt{\lambda}}P_x = \frac{\rho^2}{4\Delta_x}\dot{x}, \quad \frac{1}{\sqrt{\lambda}}P_y = \frac{\rho^2}{4\Delta_y(1-y^2)}\dot{y}$$
$$\frac{1}{\sqrt{\lambda}}P_A = \frac{\Delta_x}{\rho^2}\dot{A} \quad \frac{1}{\sqrt{\lambda}}P_B = \frac{\Delta_y(1-y^2)}{4\rho^2}\dot{B}.$$

The Hamiltonian

$$H = \frac{1}{2\sqrt{\lambda}}\left(P_{\tilde{\tau}}^2 + \frac{4\Delta_x}{\rho^2}P_x^2 + \frac{4\Delta_y(1-y^2)}{\rho^2}P_y^2 + \frac{\rho^2}{\Delta_x}P_A^2 + \frac{4\rho^2}{\Delta_y(1-y^2)}P_B^2\right). \tag{30}$$

Geodesic motion:

$$P_x = P_y = 0 \to x = x_0,\ y = y_0.$$

Thus, $P_A = P_B = 0 \to \dot{\phi} = \dot{\psi} = 0 \Rightarrow \dot{\sigma} = 0 :\ P_{\tilde{\tau}} \to P_\tau \equiv \dot{\tilde{\tau}} \quad \Longrightarrow$.

$$\boxed{\frac{P_\phi}{P_\tau} = \frac{(\alpha - x_0)(1-y_0)}{\alpha} = \frac{2P_\phi}{3P_R} + 1, \quad \frac{P_\psi}{P_\tau} = \frac{(\beta - x_0)(1+y_0)}{\beta} = \frac{2P_\psi}{3P_R} + 1}$$

Schlesinger and Heun

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- Let us have a closer look at the Fuchsian equation

$$\frac{d\Psi}{dz} = \left[\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_2}{z-t}\right]\Psi, \tag{31}$$

where, without loss of generality, the coefficient matrices $A_\nu, \nu = 0, 1, 2$, are general, and the system is diagonal at $z = \infty$, i.e.,

$$\operatorname{Tr} A_\nu = 2\theta_\nu, \quad \nu = 0, 1, 2; \qquad A_\infty = -A_0 - A_1 - A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2. \end{pmatrix} \tag{32}$$

Let us denote the eigenvalues of A_ν by

$$\pm\theta_0; \pm\theta_1, \pm\theta_2, \qquad 2\theta_0, 2\theta_1, 2\theta_2 \notin \mathbb{Z}.$$

In a compact form Schlesinger equations reads $(A_i = A_i(a_1, \ldots, a_n))$

$$\frac{\partial A_i}{\partial a_j} = (1-\delta_{ij})\frac{[A_i, A_j]}{a_i - a_j} - \delta_{ij}\sum_{k\neq i}\frac{[A_i, A_k]}{a_i - a_k};$$

- The second order ODE for the first component of $\Psi = (\psi_1\,,\,\psi_2)^T$:

$$\partial_z^2\psi_1 - (\mathrm{Tr}\,A(z) + \partial_z \log A_{12}(z))\partial_z\psi_1 + \left(\det A(z) + A_{11}(z)\partial_z \log \frac{A_{12}(z)}{A_{11}(z)}\right)\psi_1 = 0. \quad (33)$$

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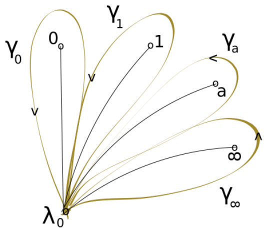

The monodromy group $\mathfrak{M}$; the base point λ_0; the branch cuts $[\lambda_0, 0]$; $[\lambda_0, 1]$; $[\lambda_0, a]$; $[\lambda_0, \infty]$ and the corresponding loops $\gamma_0, \gamma_1, \gamma_a, \gamma_\infty$. The complete monodromy data - in M_ν, $\nu = 0, 1, a, \infty$ realizing representation of $SL(2,\mathbb{Z})$ of the loops γ_ν. Conditions on monodromy matrices are:

$\det M_\nu = 1, \qquad \nu = 0, 1, a, \infty \quad M_\infty M_t M_1 M_0 = 1,$ (cyclic condition)

$$M_\infty = \begin{pmatrix} e^{2\pi i\delta} & 0 \\ 0 & e^{-2\pi i\delta} \end{pmatrix} \quad (34)$$

- Monodromy data ($M_t \equiv M_2$, $M_\infty \equiv M_3$) w/ inv. coordinates on it

$$
\begin{aligned}
a_\nu &= \mathrm{Tr}\, M_\nu = 2\cos 2\pi\alpha_\nu, \qquad \nu = 0,1,2,3 \\
t_{\mu\nu} &= \mathrm{Tr}\, M_\mu M_\nu = 2\cos\sigma_{\mu\nu}, \qquad \mu,\nu = 0,1,2.
\end{aligned}
\tag{35}
$$

- For Heun equation - take $\mathrm{tr}\, A_i = 2\theta_i$ and fix

$A_\infty = -\sum_{i=0,1,t} A_i = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$

\+ Fricke-Jimbo relation (leaves two independent t_{ij}):

$$
\begin{aligned}
W(t_{0t}, t_{1t}, t_{01}) = t_{0t}t_{1t}t_{01} + t_{0t}^2 + t_{1t}^2 + t_{01}^2 - t_{0t}(a_1 a_\infty + a_0 a_t) - t_{1t}(a_0 a_\infty + a_1 a_t) \\
- t_{01}(a_t a_\infty + a_0 a_1) + a_0^2 + a_1^2 + a_t^2 + a_\infty^2 + a_0 a_1 a_t a_\infty = 4.
\end{aligned}
$$

The relations $2\theta_\infty = \kappa_1 - \kappa_2 - 1$ and $\kappa_1 + \kappa_2 = -2(\theta_0 + \theta_1 + \theta_t)$ can be solved as

$$
\kappa_1 = \theta_\infty + \frac{1}{2} - \sum_{i=0,1,t} \theta_i, \quad \kappa_2 = -\theta_\infty - \frac{1}{2} - \sum_{i=0,1,t} \theta_i. \tag{36}
$$

$$
\mu := \sum_{i=0,1,t} \frac{p_i + 2\theta_i}{\lambda - a_i}; \quad A_{12}(z) = k\frac{z - \lambda}{z(z-1)(z-t)}, \quad k \in \mathbb{C}, \tag{37}
$$

Canonical form of deformed Heun equation

$$\partial_z^2\psi_1 + g_1(z)\partial_z\psi_1 + g_2(z)\psi_1 = 0, \tag{38a}$$

$$g_1(z) = \frac{1-2\theta_0}{z} + \frac{1-2\theta_1}{z-1} + \frac{1-2\theta_t}{z-t} - \frac{1}{z-\lambda}, \tag{38b}$$

$$g_2(z) = \frac{\kappa_1(\kappa_2+1)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{38c}$$

with the accessory parameter $K = K(\theta; x, \mu, t)$ given by

$$K(\theta;\lambda,\mu,t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \times \left[\mu^2 - \left(\frac{2\theta_0}{\lambda} + \frac{2\theta_1}{\lambda-1} + \frac{2\theta_t-1}{\lambda-t}\right)\mu + \frac{\kappa_1(\kappa_2+1)}{\lambda(\lambda-1)}\right]. \tag{39}$$

• Define

$$A(z,t) = \left[\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right]\Psi(z,x); \quad B(z,t) = -\frac{A_t}{z-t}\Psi(z,t). \qquad (40)$$

Zero-curvature cond $\partial_z A - \partial_t B - [A,B] = 0$ is satisfied if A_i satisfy Schlesinger eqs.
$\rightarrow$ Write Schlesinger for deformed Heun and parmetrize A_i as

$$A_i = \begin{pmatrix} p_i + 2\theta_i & p_i q_i \\ -\frac{(p_i+2\theta_i)}{q_i} & -p_i \end{pmatrix}, \quad A_\infty = -\sum_{i=0,1,t} A_i = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix},$$

where p_i and q_i now are functions of (λ, t) and the fixed parameters.
• Compatibility condition for (40)

$$\frac{d\lambda}{dt} = \{K, \lambda\}, \qquad \frac{d\mu}{dt} = \{K, \mu\}, \qquad (\{,\} = \partial_\mu\partial_\lambda - \partial_\lambda\partial\mu)$$

- a change of the true singularity $t \Longrightarrow$ a change in the parameters.
- μ and λ are canonically conjugated coordinates in the phase space of isomonodromic deformations.

Explicitly

$$\dot{\lambda} = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[2\mu - \left(\frac{2\theta_0}{\lambda} + \frac{2\theta_1}{\lambda-1} + \frac{2\theta_t - 1}{\lambda - t}\right)\right] \tag{41}$$

$$\dot{\mu} = \Big\{\Big[-3\lambda^2 2(1+t)\lambda - t\Big]\mu^2 + \left[2(2\lambda-1-t)\theta_0 + 2(2\lambda-t)\theta_1 \right. \\ \left. +(2\lambda-1)(2\theta_t-1)\right]\mu - \kappa_1(\kappa_2).\Big\} \tag{42}$$

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Equivalently, for λ only this is Painleve VI

$$\ddot{\lambda} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\dot{\lambda}^2 - \left(\frac{1}{t} + \frac{1}{t-1}\frac{1}{\lambda-t}\right)\dot{\lambda} \\ + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha - \gamma\frac{t}{\lambda^2} + \beta\frac{t-1}{(\lambda-1)^2} + (\frac{1}{2}-\delta)\frac{t(t-1)}{(\lambda-t)^2}\right) \tag{43}$$

where

$$\alpha = \frac{1}{2}(2\theta_\infty - 1)^2 \quad \gamma = 2\theta_0, \quad \beta = 2\theta_1^2, \quad \delta = 2\theta_t(\theta_t - 1) \tag{44}$$

- Painleve VI equation describes isomonodromy flow!

Reductions of Painleve VI

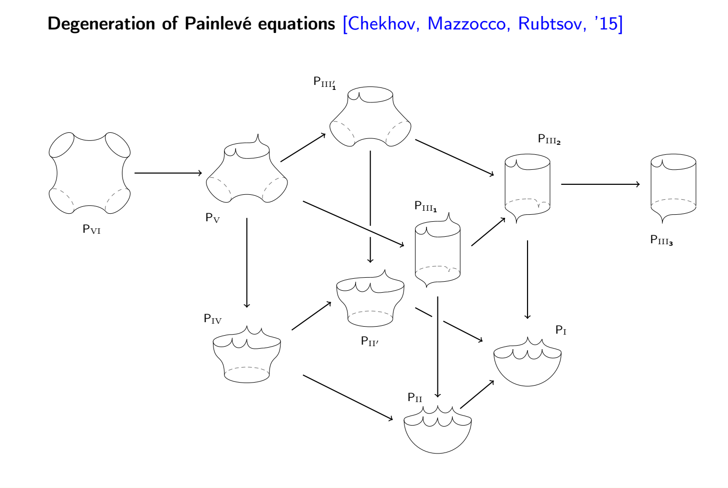
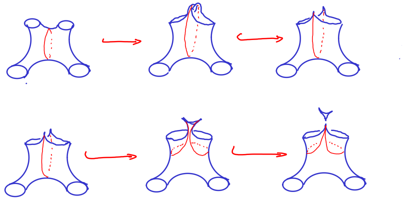


Figure: The table of confluences of Riemann surfaces from the Painlevé perspective.

• Degeneration of surfaces corresponding to reductions of Painleve equations (from [Chekhov, Mazzocco, Rubtsov 15'].)



PE w/ 4 singular points have reps in terms of Riemann surfaces. Geometric transition between different Painleve's - different types degeneration of the corresponding Riemann surfaces.

For instance, degeneration as in the first line of the figure gives

$$P_{VI} \to P_V : t \to 1 + \epsilon t_1, \quad \beta \to -\beta_1, \quad \gamma \to \delta_1 \epsilon^{-2} + \gamma_1 \epsilon^{-1}$$
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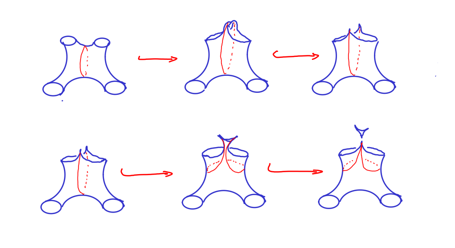

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- Functions corresponding to some surfaces

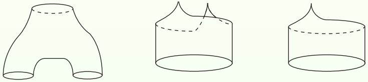

Figure: Gauss hypergeometric (3 regular punctures), Whittaker (1 regular + 1 of Poincaré rank 1) and Bessel (1 regular + 1 of rank 1/2) [Gavrilenko, Lisovyy 16'].

A side remark on Schwarz-Christoffel map and ...

Schwarz-Christoffel accessory parameters. We start with the formula of Christoffel-Schwarz mapping

$$\frac{df(w)}{dw} = \gamma \prod_{i=1}^{n} (w - w_i)^{\theta_i - 1}, \tag{45}$$

where w_i are called pre-vertices (on the line), and z_i - the pre-images of the vertices (vertices of the polygon, $z_i = f(w_i)$).

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The Schwarzian differential equation

$$\boxed{\{f(w), w\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \sum_{i=1}^{n}\left[\frac{1-\theta_i^2}{2(w-w_i)^2} + \frac{2\beta_i}{w-w_i}\right],} \tag{46}$$

where n is the number of vertices and $\pi\theta_i$ are the interior angles at each vertex z_i.

The solutions of the above equation is given by $z = f(w)$ which can be written as $\boxed{f(w) = \tilde{y}_1/\tilde{y}_2}$. Here $\tilde{y}_i$ are the two independent solutions of

$$\tilde{y}''(w) + \sum_{i=1}^{n}\left[\frac{1-\theta_i^2}{4(w-w_i)^2} + \frac{\beta_i}{w-w_i}\right]\tilde{y}(w) = 0. \tag{47}$$

Requiring that the solutions behave well at $w = \infty$ imposes algebraic constraints on the accessory parameters

$$\sum_i \beta_i = \sum_i (w_i\beta_i + 1 - \theta_i^2) = \sum_i (2w_i\beta_i^2 + w_i(1-\theta_i^2)) = 0. \tag{48}$$

By applying the transformation

$$\boxed{\tilde{y}(w) = w^{-\theta_0/2}(w-1)^{-\theta_1/2}(w-t)^{-\theta_t/2}y(w),} \tag{49}$$

we find the Heun equation in *canonical form*

$$y''(w) + \left(\frac{1-\theta_0}{w} + \frac{1-\theta_t}{w-t} + \frac{1-\theta_1}{w-1}\right)y'(w) + \left(\frac{\kappa_-\kappa_+}{w(w-1)} - \frac{t(t-1)K_0}{w(w-1)(w-t)}\right)y(w) = 0. \tag{50}$$

The constants and undefomed Hamiltonian K_0 are

$$\kappa_{\pm} = 1 - \frac{1}{2}(\theta_0 + \theta_t + \theta_1 \pm \theta_\infty) \qquad K_0 = -\beta_t + \sum_{i \neq t} \frac{(1-\theta_t)(1-\theta_i)}{2(w_i - t)}.$$

Examples of Schwarz-Christoffel maps

- The straight line passing through z_1 and z_2

$$\bar{z} = S(z) = \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} z + \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{z_1 - z_2}. \tag{51}$$

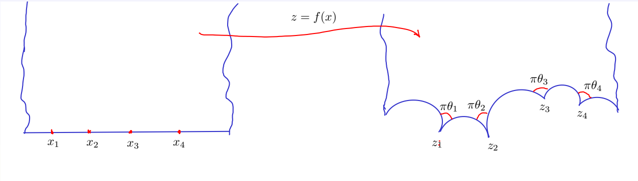
- The circle of radius r, center at z_0

$$\bar{z} = S(z) = \frac{r^2}{z - z_0} + \bar{z}_0. \tag{52}$$

- The ellipse $(z^2/a^2) + (y^2/b^2) = 1, \quad (a > b)$

$$\bar{z} = S(z) = \frac{a^2 + b^2}{a^2 - b^2} z + \frac{2ab}{a^2 - b^2} \sqrt{z^2 + b^2 - a^2}. \tag{53}$$

As a map from UHP to a polycircular-shaped domain



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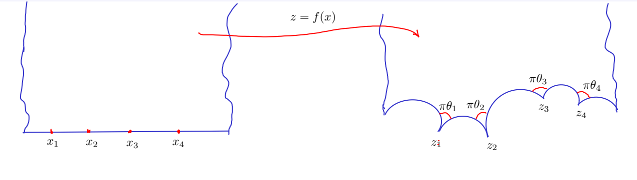
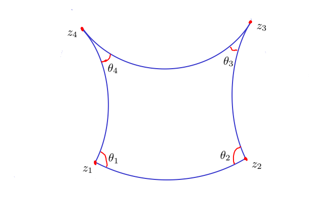


Schwarz-Christoffell graph



For $f(w) = y_1(w)/y_2(w)$

$$\bar{z} = S_i(z) = \frac{\bar{x}_i z + r_i^2 - |x|^2}{z - x_i}.$$

The centers of circle arcs C_i: x_i; radius: r_i; angles: $\pi\theta_i$.

In terms of the single monodromy parameters ($M_i = S_{i+1}\bar{S}_i$)

$$2\cos\theta_i = \frac{x_i\bar{x}_{i+1} + r_i^2 - |x_i|^2 + \bar{x}_i x_{i+1} + r_{i+1}^2 - |x_{i+1}|^2}{r_i r_{i+1}}.$$

$\Longrightarrow$ Schwarz-Christoffel graph is built out from the single monodromy parameters.

(Non)integrability issues

- For PVI non-integrability:
  Theorem 1. Let $\theta_\infty = \theta_1 + \theta_2 + \theta_t$ and at least one $\theta_j \in \mathbb{Z}$ and at least one $\theta_k \notin \mathbb{Q}$. Then the sixth Painleve equation is not integrable.
  Theorem 2. Let $\theta_\infty = \theta_1 + \theta_2 + \theta_t$ and at least two θ_j are integers. Then the sixth Painleve equation is not integrable.

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- For non-integrability of strings in $Y^{p,q}$ background:
  - Basu & Pando Zayas 11' considered $Y^{p,q}$ with the simplest ansatz

$$\theta = \theta(\tau), \quad \mu = \mu(\tau), \quad y = y(\tau), \quad \phi = \alpha_1 \sigma, \quad \psi = \alpha_2 \sigma. \tag{54}$$

- $\dot{\theta}(\tau) = \theta(\tau) = 0$ solves string EoM.
- for remaining y-eq

$$\ddot{y} - \frac{p'}{p}\dot{y}^2 + \frac{p\,p'}{2}(\alpha_2 + c\,\alpha_1)^2 + \frac{2}{3}p(\alpha_2 + c\,\alpha_1)(y(\alpha_2 + c\,\alpha_1) - \alpha_1) = 0. \tag{55}$$

- the Normal Variational Equation takes the form

$$\ddot{\eta} - \frac{c\,\dot{y}_s}{1 - cy_s}\dot{\eta} + \alpha_1 \left(\alpha_1 - \frac{c\,p(y_s)}{1 - c\,y_s}(\alpha_2 + c\,\alpha_1) - \frac{2}{3}((\alpha_2 + c\,\alpha_1)y_s - \alpha_1) \right) \eta = 0$$

(Non)integrability issues

- wrining Normal Variational Equation in appropriate form and applying systematically Kovacic' algorithm, it fails to yield a solution pointing that the system is generically non-integrable.
- consider the simpler geometry $T^{1,1}$

$$ds^2 = R^2 \Big(-\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega_3^2 + \frac{1}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2) + \frac{1}{9}(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i)^2 \Big) . \quad (56)$$

with tha ansatz

$$\phi_1 = \alpha_1\sigma, \quad \phi_2 = \alpha_2\sigma, \quad t = t(\tau), \quad \psi = \psi(\tau), \quad \theta_i = \theta_i(\tau).$$

$\Longrightarrow$ Kovacic' algorithm fails again for generic values of constants.

- For these solutions, we found that the condition for firts theorem for non-integrability of Painleve VI is satisfied!
- **Conjecture**: There exist correspondence between string non-integrability in strings in $Y^{p,q}$ background and PVI non-integrability.

Other issues

- Different SE backgrounds $\rightarrow$ different Heun equation $\rightarrow$ Painleve equations $\rightarrow$ different singularity structures

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• **Conjecture**: Confluent limits of Painleve VI encode the changes of background geometry.

Again: the confluent limit PVI $\rightarrow$ PV

$$P_{VI} \rightarrow P_V : t \rightarrow 1 + \epsilon t_1, \quad \beta \rightarrow -\beta_1, \quad \gamma \rightarrow \delta_1 \epsilon^{-2} + \gamma_1 \epsilon^{-1}$$
$$\delta \rightarrow -\delta_1 \epsilon^{-2}, \quad (\epsilon \rightarrow 0)$$

The corresponding (deformed) confluent Heun equation is

$$y''(z) + \left[\frac{1-2\tilde{\theta}_0}{z} + \frac{1-2\tilde{\theta}_t}{z-t} - \frac{1}{z-\lambda}\right] y'(z) + \left[-\frac{1}{4} + \frac{2\tilde{\theta}_\infty - 1}{2z} - \frac{tc}{z(z-t)} + \frac{\lambda\mu}{z(z-\lambda)}\right] y(z) = 0.$$

Thus

$$t = \frac{1}{2}\left(1 + \frac{\sqrt{4p^2 - 3q^2}}{q}\right) \longrightarrow 1 \quad \Longrightarrow \quad Y^{p,q} \longrightarrow T^{p,p} \; (T^{1,1})$$

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Future directions:

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Future directions:

\- complete spectrum of Sasaki-Eistein backgrounds

\- (black hole) backgrounds w/ monodromies associaed to other Painleve's

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\- Seiberg-Witten curves?

\- ...

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THANK YOU!