Open quantum systems and (non)Markovianity

BWAM22

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September 2022

- Real quantum systems can not be described solely by Schrodinger equation

$$\imath\hbar\frac{d|\Psi(t)\rangle}{dt} = \hat{H}(t)|\Psi(t)\rangle \tag{1}$$

which is dynamical law for isolated systems, with standard notation.

- On the contrary, real quantum systems (S) are open (1). This means unavoidable interaction with environment (E). The whole S+E makes isolated system governed by eq.(1). Dynamics of (sub)system is now described with

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{\imath}{\hbar}[\hat{H}_S, \hat{\rho}(t)] + \int_{t_\circ}^{t} dr\mathcal{K}(t,r)\hat{\rho}(r) \tag{2}$$

where $\hat{H}_S$ is the open-system's self-Hamiltonian, while the linear map $\mathcal{K}(t,r)$ is the "memory kernel" describing the effects of the environment on the system.

- Above description of dynamics is in a differential form for both cases. As is known, dynamics can be written in integral form. Namely, for isolated systems such is the whole $S+E$ dynamics is described by unitary operator $\hat{U}(t,t_\circ)$ which is equivalent to eq.(1). The open system's dynamics is determined by the time-dependent reduced density matrix $\hat{\rho}_S(t) = tr_E\left(\hat{U}(t,t_\circ)\hat{\rho}_{SE}(t_\circ)\hat{U}^\dagger(t,t_\circ)\right)$ with the total $S+E$ system's state $\hat{\rho}_{SE}(t_\circ)$ and the initial time instant $t_\circ$.
- Subsystem dynamics can then be written via dynamical map $\hat{\rho}(t) = \Phi(t,t_\circ)\hat{\rho}(t_\circ)$ or in other words: map $\Phi(t,s)$ governs a state transition, $\hat{\rho}(s) \rightarrow \hat{\rho}(t)$.

- In quantum physics (and their applications) the maps with the following features are of interest: **linear, differentiable, decomposable** and **positive** maps. Beside this, **invertibility** and **time locality** of maps are desirable.
- Also, it is required stronger property than just positivity of the map: complete positivity. In other words, map is valid physical map if $\Phi(t, t_\circ) \otimes Id$ is positive, where Id stands for identical map (operator) on the Hilbert space of arbitrary dimension. This property of quantum dynamics comes from physical reality of unavoidable existence of quantum correlations of open system with environment. Clearly, complete positivity implies positivity, but not the other way around.

- As a topic of special interest and a vivid ongoing research appear the so-called **Markovian** (as opposite to **non-Markovian**) dynamical maps (dynamical processes).
- Markovian processes are well known in classical physics. Essentially, a Markov process is a stochastic process $X(t)$ with a short memory, that is a process which rapidly forgets its past history. This property is what makes a Markov process so easy to deal with, since it ensures that the whole hierarchy of joint probabilities can be reconstructed from just two distribution functions.
- In quantum physics Markovian paradigm is of considerable interest, but we should have in mind underlaying non-commutative algebra as a distinction from classical physics. As a consequence, it is notoriously hard to formulate a proper quantum-mechanical counterpart of the classical concept of Markovianity of dynamical maps.

- In general, quantum processes (dynamics) are non-Markovian. From the practical point of view, presence of some kind of memory in the open system's dynamics may be dangerous or even fatal for certain purposes and applications, e.g. in the nascent field of quantum technology, notably in quantum information processing and quantum computation (2), and quantum metrology (3).
- To proceed further, an definition is in order

Definition (Markovian quantum process)

We shall say that a quantum system subject to a time evolution given by some family of trace-preserving linear maps $\{\Phi_{(t_2,t_1)}, t_2 \geq t_1 \geq t_0\}$ is *Markovian* (or *divisible* (4)) if, for every t_2 and t_1, $\Phi_{(t_2,t_1)}$ is a complete positive map and fulfills the composition law $\Phi_{(t_3,t_1)} = \Phi_{(t_3,t_2)}\Phi_{(t_2,t_1)}, \quad t_3 \geq t_2 \geq t_1$.

- The above Definition is about quantum maps and their Markovianity. If quantum map is differentiable and admits master equation as a dynamical law, Markovianity can be expressed and described by the following theorem (1,5)

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad)

An operator $\mathcal{L}_t$ is the generator of a quantum Markov (or divisible) process if and only if it can be written in the form

$$\frac{d\rho(t)}{dt} = \mathcal{L}_t[\rho(t)] = -\mathrm{i}[H(t), \rho(t)] + \sum_k \gamma_k(t) \left[V_k(t)\rho(t)V_k^\dagger(t) - \frac{1}{2}\{V_k^\dagger(t)V_k(t), \rho(t)\} \right], \tag{3}$$

where $H(t)$ and $V_k(t)$ are time-dependent operators, with $H(t)$ self-adjoint, and $\gamma_k(t) \geq 0$ for every k and time t.

- Is there any way to distinguish between Markovian and non-Markovian maps? During the paste ten years a number of witnesses have been constructed. For example, BLP criterion is based on derivative of trace distance and then optimization of the proper integral over the time of quantum evolution. RHP criterion, on the other hand, is based on the property of map divisibility, with request that intermediate map be completely positive, which can be checked with quite complicated algorithm. The two criteria are known not to be mutually equivalent (6, 7): while RHP imply the BLP, the reverse is, in general, not true.
- There are other criteria of Markovianity and more or less they are linked with some geometrical property of Hilbert space or need for optimization, which makes them hard for implementation. All this suggests that there is no canonical criteria for non-Markovianity.

- Our starting point is a *common point of agreement* of some important criteria of Markovianity. Assume that a dynamical process admits a master equation (differential) form for the open system's reduced state $\hat{\rho}_S(t)$. Then the mentioned common point is the requirement that in order for the process to be Markov, it *must* be *local in time*:

$$\frac{d\hat{\rho}_S(t)}{dt} = \mathcal{L}_t \hat{\rho}_S(t). \tag{4}$$

- Locality of $\mathcal{L}_t$ is a *necessary condition* for a process to be regarded Markovian. In other words: a (time-)differentiable dynamical process not admitting a master equation of the general form of eq.(4) is necessarily *non-Markovian*.
- But, can not be overemphasized: eq.(4) applies also to some non-Markovian processes. That is, time-locality of a master equation is necessary but not sufficient for Markovianity of the process.

- Natural assumptions: Basic physical laws are typically expected to be linear, continuous and smooth in time thus providing a differential mathematical form, i.e. a related differential equation whose solutions sufficiently describe dynamics and behavior of physical systems.
- Introduction of so called $\mathcal{C}$ class of dynamical maps.

Definition ($\mathcal{C}$ class)

A *linear* and *completely positive* dynamical map Φ is in the so-called $\mathcal{C}$ class of dynamical maps if and only if the following requirements are simultaneously fulfilled: (a) the map is time continuous, in the sense it is defined on a continuous time interval $t' \in [t_\circ, t]$, (b) the map is a two-parameter map denoted $\Phi(t, t_\circ), t \geq t_\circ$, (c) the map is smooth enough (ultraweak continuity), in the sense that, for positive ϵ, $\lim_{\epsilon \downarrow 0} \Phi(t+\epsilon, t_\circ) = \Phi(t, t_\circ)$, for $t \geq t_\circ$, is well defined, (d) the map has the whole Banach space of statistical operators (density matrices) in its domain, and (e) the map is differentiable, i.e. that the (ultraweak) limit:

$$\frac{d\Phi(t, t_\circ)}{dt} = \lim_{\epsilon \downarrow 0} \frac{\Phi(t+\epsilon, t_\circ) - \Phi(t, t_\circ)}{\epsilon} \tag{5}$$

is well defined.

- Every dynamical map that is not linear or non-completely positive or not satisfying at least some of the above conditions (a)-(e) of Definition does not belong to the $\mathcal{C}$ class of dynamical maps.
- The central result is the following Lemma

Lemma

1. For a dynamical map $\Phi(t, t_\circ)$ *from the* $\mathcal{C}$ *class* of dynamical maps, the following characteristics of the map are mutually *equivalent*: (i) the map is invertible, (ii) the map is divisible, and (iii) the map admits a time-local master equation.

Proof.

We provide a proof of the lemma by establishing the chain of implications:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) \tag{6}$$

Proof.

$(i) \Rightarrow (ii)$: Assuming existence of the inverse map, $\Phi^{-1}(t, t_\circ)$, it easily follows $\hat{\rho}(t) = \Phi(t, t_\circ)\hat{\rho}(t_\circ) = \Phi(t, t_\circ)\Phi^{-1}(s, t_\circ)\hat{\rho}(s), t \geq s \geq t_\circ$. This expression presents a state transition, $\hat{\rho}(s) \to \hat{\rho}(t)$, and introduces the map $\Phi(t, s)$ for this transition. The requirement that everything regards arbitrary initial state $\hat{\rho}(t_\circ)$ implies divisibility of the map: $\Phi(t, s) = \Phi(t, t_\circ)\Phi^{-1}(s, t_\circ)$;
$(ii) \Rightarrow (iii)$: Assuming divisibility of the map, leads (ultraweak continuity) to:

$$\frac{d\hat{\rho}(t)}{dt} = \lim_{\epsilon \to 0} \frac{\Phi(t+\epsilon, t_\circ) - \Phi(t, t_\circ)}{\epsilon}\hat{\rho}(t_\circ) = \lim_{\epsilon \to 0} \frac{\Phi(t+\epsilon, t) - \mathcal{I}}{\epsilon}\hat{\rho}(t) := \mathcal{L}_t\hat{\rho}(t) \tag{7}$$

which is a time-local master equation describing dynamics generated by the time-local Liouvillian $\mathcal{L}_t := \lim_{\epsilon \to 0} \frac{\Phi(t+\epsilon, t) - \mathcal{I}}{\epsilon}$;
$(iii) \Rightarrow (i)$: The map $\Phi(t, t_\circ)$ can be presented due to the so-called time-splitting formula in the form:

$$\Phi(t, t_\circ) = \lim_{\max|t'_{j+1} - t'_j| \to 0} \Pi^0_{j=n-1} e^{\mathcal{L}_{t'_j}(t'_{j+1} - t'_j)}, \tag{8}$$

Proof.

where $t = t'_n \geq t'_{n-1} \geq \cdots \geq t_\circ$ and $\mathcal{L}_t$ is the Liouvillian. Then the inverse is constructed:

$$\Phi^{-1}(t, t_\circ) = \lim_{\max|t'_{j+1} - t'_j| \to 0} \Pi_0^{j=n-1} e^{-\mathcal{L}_{t'_j}(t'_{j+1} - t'_j)}, \tag{9}$$

as it can be easily seen by inspection. □

- Typically, invertibility is assumed to be non-essential for Markovianity of the open-quantum-system dynamical maps.
- Lemma establishes invertibility as a *witness* of Markovianity: as distinguished above, non-invertibility, equivalently, non-divisibility, implies time non-locality and therefore non-Markovian character for the $\mathcal{C}$-class dynamical maps.

- Some features should be highlighted: (A) the product in equation (8) (as well as in equation (9)) assumes the time-ordering thus providing the solution to equation (4) in the standard exponential form, $\Phi(t, t_\circ) = \mathcal{T} \exp\left(\int_{t_\circ}^{t} \mathcal{L}(s) ds\right)$, which is formally often used even for the continuous-variable systems (1,8), (B) the proof of Lemma is **exact**, i.e. it may not apply for certain approximation methods (e.g. perturbative approximation of Liouvillian) or short-time behavior (while bearing in mind that for sufficiently short time-intervals, all dynamical maps are (approximately) invertible) (1,8)

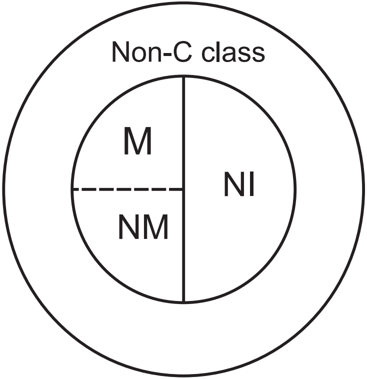

Figure: A schematic presentation of the dynamical maps with the internal circle containing all the $\mathcal{C}$-class dynamical maps sharply divided from the non-$\mathcal{C}$-class dynamical maps (that are out of the internal circle). The vertical solid line sharply divides invertible (the left part) from non-invertible (time-nonlocal, i.e. indivisible, the right part) $\mathcal{C}$-class processes. The dashed horizontal line sharply divides Markovian (upper part) from the non-Markovian–invertible $\mathcal{C}$-class–processes. Position of the dashed line is not yet uniquely determined–it depends on the adopted definition (criterion) of Markovianity. The used abbreviations are as follows: "Non-C class" stands for the non-$\mathcal{C}$-class processes; "NI" stands for "noninvertible" $\mathcal{C}$-class processes; "M" is for "Markovian" while "NM" is for "non-Markovian" (invertible) $\mathcal{C}$-class processes.

- Fortunately, invertibility of a dynamical map is straightforward operationally to test. Concretely, as distinct from most of the existing Markovianity witnesses in the literature, testing invertibility does not operationally require any kind of optimization.
- For the $\mathcal{C}$-class processes, complete state tomography (2) *suffices* for determination of non-Markovian character of the process.
- **In conclusion** $\mathcal{C}$ class introduced by Definition is inspired by the fact that all basic physical laws are in the $\mathcal{C}$ class of dynamical maps, notably Newton's second law, the Hamilton's and Lagrange's equations of classical mechanics, the Maxwell equations of classical electrodynamics as well as the time-dependent Schrödinger equation. That is, the basic physical laws are expected to be continuous in time as well as differentiable while free of any singularities for every finite time instant $t' \in [t_\circ, t]$.

- This leads to invertibility as a criterion with simple operational meaning and use for distinguishing Markovian and non-Markovian dynamics.
- Dropping any of conditions which defines $\mathcal{C}$ class of dynamical processes leads, on the other hand, beyond that class and can be questioned in light of natural assumptions mentioned above.
- This presentation is based on the work in progress under title: "Invertibility as a witness of Markovianity of the quantum dynamical maps" (J. Jeknić-Dugić, M. Arsenijević and M. Dugić).

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Thanks for attention