# Dynamical approximations in multifield cosmological models

**Calin Lazaroiu** **NIPNE, Romania**

# Outline

**Definition**

An n-dimensional scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, \Phi)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a connected, borderless and complete Riemannian *n*-manifold (called scalar manifold)
- $\Phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R}_{>0})$ is a smooth strictly positive function (called scalar potential).

A cosmological model is a system $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ where $(\mathcal{M}, \mathcal{G}, \Phi)$ is a scalar triple and the parameter $M_0 > 0$ is called *rescaled Planck mass*.

**Definition**

The rescaled Hubble function $\mathcal{H} : T\mathcal{M} \to \mathbb{R}_{>0}$ of a cosmological model $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ is defined through:

$$\mathcal{H}(u) \stackrel{\text{def.}}{=} \frac{1}{M_0}\sqrt{||u||^2 + 2\Phi(\pi(u))} \quad \forall u \in T\mathcal{M}$$

where $\pi : T\mathcal{M} \to \mathcal{M}$ is the bundle projection. The cosmological equation of $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ is the autonomous geometric second order ODE:

$$\nabla_t \dot{\varphi}(t) + \mathcal{H}(\dot{\varphi}(t))\dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}}\Phi)(\varphi(t)) = 0 \quad .$$

**Definition**

The cosmological semispray of $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ is the defining smooth second order vector field $S \in \mathcal{X}(T\mathcal{M})$. The pair $(T\mathcal{M}, S)$ is called the cosmological dynamical system of $(\mathcal{M}, \mathcal{G}, \Phi)$ at rescaled Planck mass $M_0$. The flow $\Pi \in \mathcal{C}^\infty(\mathcal{D}, T\mathcal{M})$ (with $\mathcal{D} \subset \mathbb{R} \times T\mathcal{M}$) of this dynamical system is called the cosmological flow of $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$.

**Definition**

The solutions $\varphi : I \to \mathcal{M}$ of the cosmological equation of $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ (where $I$ is a non-degenerate interval) are called cosmological curves, while their canonical lifts $\gamma \stackrel{\text{def.}}{=} \dot{\varphi} : I \to T\mathcal{M}$ are called cosmological *flow* curves. The latter are the integral curves of $S$.

**Definition (Critical and non-critical sets)**

$$\mathrm{Crit}\Phi \stackrel{\text{def.}}{=} \{m \in \mathcal{M} \mid (\mathrm{d}\Phi)(m) = 0\} \ \ , \ \ \mathcal{M}_0 \stackrel{\text{def.}}{=} \mathcal{M} \setminus \mathrm{Crit}\Phi$$

**Definition**

For any curve $\varphi : I \to \mathcal{M}$, define $I_{\mathrm{reg}} \stackrel{\text{def.}}{=} \{t \in I \mid \dot{\varphi}(t) \neq 0\}$.

Each scalar triple defines a cosmological model on $\mathbb{R}^4$:

$$\mathcal{S}_{\mathcal{M},\mathcal{G},\Phi}[g,\varphi] = \int_{\mathbb{R}^4} \mathrm{d}^4x\,\sqrt{|g|}\left[\frac{M^2}{2}R(g) - \frac{1}{2}\mathrm{Tr}_g\varphi^*(\mathcal{G}) - \Phi\circ\varphi\right] .$$

Define the *rescaled Planck mass* through $M_0 \stackrel{\mathrm{def.}}{=} \sqrt{\frac{2}{3}}M$, where $M$ is the reduced Planck mass. Take $g$ to describe a spatially flat FLRW universe:

$$\mathrm{d}s_g^2 := -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2 \;\; (x^0 = t \;\; , \;\; \vec{x} = (x^1, x^2, x^3) \;\; , \;\; a(t) > 0 \;\forall t)$$

and $\varphi$ to depend only on the cosmological time $\varphi = \varphi(t)$. Define the *Hubble parameter* through $H(t) \stackrel{\mathrm{def.}}{=} \frac{\dot{a}(t)}{a(t)}$. When $H > 0$, the equations of motion are equivalent with the cosmological equation of $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ together with the condition:

$$H(t) = \frac{1}{3}\mathcal{H}(\dot{\varphi}(t)) \;\; ,$$

which determines $a$ up to an integration constant along each cosmological curve.

**Definition**

A *regular dynamical approximant* of the cosmological semispray $S$ of $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ is a smooth second order vector field $S_0 \in \mathcal{X}(T\mathcal{M})$ which depends only on $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$. The gometric second order ODE defined on $\mathcal{M}$ by $S_0$ is called a *regular approximant* of the cosmological equation of $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$.

A second order dynamical approximation defines an flow $\Pi_0 : \mathcal{D}_0 \subset \mathbb{R} \times T\mathcal{M} \to \mathcal{M}$ which can be viewed as an approximant of $\Pi$ on the open set $\mathcal{D} \cap \mathcal{D}_0 \subset \mathbb{R} \times T\mathcal{M}$ in any of the the $C^k$ weak Whitney (a.k.a. compact-open) topologies of $\mathcal{C}^\infty(\mathcal{D} \cap \mathcal{D}_0, \mathcal{M})$.

**Remark**

One can also consider *denenerate approximants* of $S$, which are defined by pairs $(\mathcal{N}, S_0)$, where $\mathcal{N}$ is a section of $T\mathcal{M}$ and $S_0$ is a smooth vector field defined on $\mathcal{M}$. The latter describe first order geometric ODEs on $\mathcal{M}$. In this case, the $\pi$-pullback of $S_0$ to $\mathcal{N}$ approximates the restriction of $S$ to $\mathcal{N}$.

A natural class of dynamical approximants can be constructed using *basic scalar cosmological observables*, which are functions $f : U \to \mathbb{R}$ with $U$ an open subset of $T\mathcal{M}$.

**Definition**

The logarithmic and characteristic forms of the model $(M_0, \mathcal{M}, \mathcal{G}, \Phi)$ are the smooth exact 1-form $\Xi, \Psi \in \Omega^1(\mathcal{M})$ defined through:

$$\Xi \stackrel{\text{def.}}{=} \frac{M_0}{2} \mathrm{d} \log \Phi \quad , \quad \Psi \stackrel{\text{def.}}{=} -\mathrm{d}\left(\frac{M_0}{\sqrt{2\Phi}}\right) = \frac{\Xi}{\sqrt{2\Phi}} \quad .$$

Let $F = \pi^*(T\mathcal{M}) \to T\mathcal{M}$ be the Finsler bundle of $\mathcal{M}$.

**Definition**

The first IR function and rescaled first slow roll function $\kappa, \hat{\epsilon} : T\mathcal{M} \to \mathbb{R}_{\geq 0}$ are:

$$\kappa(u) \stackrel{\text{def.}}{=} \frac{||u||^2}{2\Phi(\pi(u))} \quad , \quad \hat{\varepsilon}(u) \stackrel{\text{def.}}{=} \frac{\kappa(u)}{1+\kappa(u)} \quad \forall u \in T\mathcal{M} \quad .$$

The relative gradient field and rescaled acceleration field $q, \hat{\eta} \in \Gamma(\dot{T}\mathcal{M}, F)$ are:

$$q(u) \stackrel{\text{def.}}{=} \frac{(\mathrm{grad}\Phi)^v(u)}{\mathcal{H}(u)||u||} \quad , \quad \hat{\eta}(u) \stackrel{\text{def.}}{=} \nu(u) + q(u)$$

where $n(u) \stackrel{\text{def.}}{=} \frac{(\mathrm{grad}\Phi)(\pi(u))}{||(\mathrm{d}\Phi)(\pi(u))||}$ and $\nu(u) \stackrel{\text{def.}}{=} \frac{u}{||u||}$ for $u \in \dot{T}\mathcal{M}$.

# Fundamental basic observables

We extend $||q||$ to $T\mathcal{M}_0$ by setting $||q(u)|| = +\infty$ when $||u|| = 0$.

**Definition**

The conservative function $c : T\mathcal{M}_0 \to \mathbb{R}_{\geq 0}$ of $(\mathcal{M}, \mathcal{G}, \Phi)$ is:

$$c(u) \stackrel{\text{def.}}{=} \frac{1}{||q(u)||} \quad \forall u \in T\mathcal{M}_0 \quad .$$

**Proposition**

*Suppose that $||\Xi||$ is known. Then $\kappa$ and $c$ are related on $T\mathcal{M}_0$ through:*

$$c(u) = \frac{[\kappa(u)(1+\kappa(u))]^{1/2}}{||\Xi(\pi(u))||} \quad , \quad \kappa(u) = \frac{1}{2}\left[-1 + \sqrt{1 + 4c(u)^2 ||\Xi(\pi(u))||^2}\right] \quad .$$

**Definition**

The rescaled second slow roll function and characteristic angle function $\hat{\eta}^{||} : \dot{T}\mathcal{M} \to \mathbb{R}$, $\Theta : \dot{T}\mathcal{M}_0 \to [0, \pi]$ are:

$$\hat{\eta}^{||}(u) = \mathcal{G}(\hat{\eta}(u), \nu(u)) =_{\dot{T}\mathcal{M}_0} 1 + \frac{\cos\Theta(u)}{c(u)} \quad , \quad \cos\Theta(u) \stackrel{\text{def.}}{=} \mathcal{G}_{\pi(u)}(\nu(u), n(u)) \quad .$$

We have $\hat{\eta}^{\parallel}(u) \in \left[1 - \frac{1}{c(u)}, 1 + \frac{1}{c(u)}\right]$.

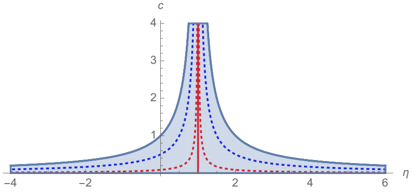


Figure: Admissible domain of $c$ and $\hat{\eta}^{\parallel}$. The right and left boundaries of the domain are the hyperbolas $c(\hat{\eta}^{\parallel} - 1) = -1$ and $c(\hat{\eta}^{\parallel} - 1) = +1$, which correspond respectively to $\Theta = 0$ and $\Theta = \pi$. The vertical red line in the middle has equation $\hat{\eta}^{\parallel} = 1$ and corresponds to $\Theta = \pi/2$. The interval within which $\hat{\eta}^{\parallel}$ can vary for a fixed value of $c$ is obtained by intersecting the corresponding horizontal line with the domain shown in the figure. The strongly dissipative regime $c \gg 1$ forces $\hat{\eta}^{\parallel}$ to be close to one and hence the ultra slow roll approximation is accurate in this regime.

**Definition**

The rescaled first slow roll parameter, rescaled acceleration and rescaled second slow roll parameter of a curve $\varphi : I \to \mathcal{M}$ are the maps $\hat{\varepsilon}_\varphi : I \to \mathbb{R}$, $\hat{\eta}_\varphi : I_{\mathrm{reg}} \to T\mathcal{M}$ and $\hat{\eta}^{\parallel}_\varphi : I_{\mathrm{reg}} \to \mathbb{R}$ defined through:

$$\hat{\varepsilon}_\varphi(t) \stackrel{\mathrm{def.}}{=} -\frac{\dot{\mathcal{H}}_\varphi(t)}{\mathcal{H}_\varphi(t)^2} = -\frac{1}{\mathcal{H}_\varphi(t)}\frac{\mathrm{d}}{\mathrm{d}t}\log \mathcal{H}_\varphi(t) \quad .$$

$$\hat{\eta}_\varphi(t) \stackrel{\mathrm{def.}}{=} -\frac{1}{\mathcal{H}_\varphi(t)}\frac{\nabla_t \dot{\varphi}(t)}{||\dot{\varphi}(t)||} \quad , \quad \hat{\eta}^{\parallel}_\varphi(t) \stackrel{\mathrm{def.}}{=} \mathcal{G}(\hat{\eta}_\varphi(t), T_\varphi(t)) \quad ,$$

where $T_\varphi(t)$ is the unit tangent vector to $\varphi$ at $t \in I_{\mathrm{reg}}$.

**Proposition**

*Suppose that $\varphi : I \to \mathcal{M}$ is a cosmological curve. Then:*

$$\hat{\varepsilon}_\varphi(t) = \hat{\varepsilon}(\dot{\varphi}(t)) \quad \forall t \in I$$

*and*

$$\hat{\eta}_\varphi(t) = \hat{\eta}(\dot{\varphi}(t)) \quad , \quad \hat{\eta}^{\parallel}_\varphi(t) = \hat{\eta}^{\parallel}(\dot{\varphi}(t)) \quad \forall t \in I_{\mathrm{reg}} \quad .$$

Write the cosmological equation as:

$$\nabla_t \dot{\varphi}(t) + \frac{1}{M_0}\sqrt{2\Phi(\varphi(t))}\left[1+\kappa_\varphi(t)\right]^{1/2}\dot{\varphi}(t) + (\mathrm{grad}_\mathcal{G}\Phi)(\varphi(t)) = 0 \quad .$$

The first dynamical slow roll approximation consists of neglecting $\kappa_\varphi$, i.e. approximating $\varphi$ by the solution $\varphi_s$ of the slow cosmological equation:

$$\nabla_t \dot{\varphi}_s(t) + \frac{1}{M_0}\sqrt{2\Phi(\varphi_s(t))}\dot{\varphi}_s(t) + (\mathrm{grad}_\mathcal{G}\Phi)(\varphi_s(t)) = 0$$

which satisfies the initial conditions:

$$\varphi_s(0) = \varphi(0) \;\; \mathrm{and} \;\; \dot{\varphi}_s(0) = \dot{\varphi}(0) \quad .$$

This is accurate when $\kappa_\varphi(0) \ll 1$, which amounts to the condition:

$$\kappa(\varphi(0)) = \kappa(\varphi_s(0)) = \frac{||\dot{\varphi}(0)||^2}{2\Phi(\varphi(0))} \ll 1 \quad .$$

A necessary condition for the approximation to be accurate at $t \neq 0$ is that:

$$\kappa(\dot{\varphi}_s(t)) \ll 1 \Longleftrightarrow ||\dot{\varphi}(t)|| \ll \sqrt{2\Phi(\varphi(t))} \quad ,$$

where:

$$\kappa(\dot{\varphi}_s(t)) = \frac{||\dot{\varphi}_s(t)||^2}{2\Phi(\varphi_s(t))} \quad .$$

Write the cosmological equation as:

$$\mathcal{H}_\varphi(t)(1-\hat{\eta}^{\|}_\varphi(t))\dot{\varphi}(t) - ||\dot{\varphi}(t)||\hat{\eta}^{\perp}_\varphi(t) + (\mathrm{grad}\Phi)(\varphi(t)) = 0 \quad .$$

The second dynamical slow roll approximation neglects $\hat{\eta}^{\|}(t)$, i.e. the quantity:

$$[\nabla_t\dot{\varphi}(t)]^{\|} = \mathcal{G}(\nabla_t\dot{\varphi}(t), T_\varphi(t)) = \frac{\mathrm{d}}{\mathrm{d}t}||\dot{\varphi}(t)|| \quad .$$

This replaces $\varphi$ by the solution $\varphi_\sigma$ of the no second roll equation:

$$[\nabla_t\dot{\varphi}_\sigma(t)]^{\perp} + \mathcal{H}_{\varphi_\sigma}(t)\dot{\varphi}_\sigma(t) + (\mathrm{grad}\Phi)(\varphi_\sigma(t)) = 0$$

which satisfies the initial conditions:

$$\varphi_\sigma(0) = \varphi(0) \ \text{ and } \ \varphi_\sigma(t) = \varphi(t) \quad .$$

We have:

$$[\nabla_t\dot{\varphi}_\sigma(t)]^{\perp} = \nabla_t\dot{\varphi}_\sigma(t) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\log||\dot{\varphi}_\sigma(t)||\right)\dot{\varphi}_\sigma(t) \quad .$$

When $\dot{\varphi}(t) = 0$, we define $[\nabla_t\dot{\varphi}(t)]^{\perp} \stackrel{\mathrm{def.}}{=} \nabla_t\dot{\varphi}(t)$. The approximation is accurate when $|\hat{\eta}^{\|}(\varphi(t))| \ll 1$, which implies $\hat{\eta}^{\|}(\varphi_\sigma(t)) \ll 1$.

# Approximations controlled by the norm of $\hat{\eta}_\varphi(t)$

Consider the approximants obtained by requiring that $||\hat{\eta}_\varphi(t)||$ is very small, very large or close to one:

- The **gradient flow condition** $||\eta_\varphi(t)|| \ll 1$ implies the second slow roll condition $|\hat{\eta}_\varphi^{||}(t)| \ll 1$ and leads to the gradient flow approximation. This degenerate dynamical approximation consists of replacing the cosmological equation with the modified gradient flow equation:

$$\mathcal{H}_\varphi(t)\dot{\varphi}(t) + (\mathrm{grad}\Phi)(\varphi(t)) = 0 \quad ,$$

  whose integral curves are reparameterized gradient flow curves of $\Phi$. Combining the gradient flow and first slow roll approximations produces the IR approximation, which is a specialization of the second order slow roll approximation and plays a crucial role in the dynamical RG flow analysis of cosmological models.
- The condition $||\hat{\eta}_\varphi(t)|| \gg 1$ is equivalent with the **conservative condition** $c(\dot{\varphi}(t)) \ll 1$, which forces $||\hat{\eta}_\varphi(t)|| \approx \frac{1}{c(\dot{\varphi}(t))}$ and leads to the conservative approximation.
- The condition $||\hat{\eta}_\varphi(t)|| \approx 1$ is equivalent with the **dissipative condition** $c(\dot{\varphi}(t)) \gg 1$, which leads to the dissipative approximation.

The conservative approximation considers only non-critical cosmological curves $\varphi : I \to \mathcal{M}_0$ (take $0 \in I$) and neglects the friction term, thus approximating $\varphi$ for small $|t|$ by the solution $\varphi_c : I_c \to \mathcal{M}$ of the conservative equation of $(\mathcal{M}, \mathcal{G}, \Phi)$:

$$\nabla_t \dot{\varphi}_c(t) + (\mathrm{grad}\Phi)(\varphi_c(t)) = 0 \quad \text{with} \quad \varphi_c(0) = \varphi(0) \quad \text{and} \quad \dot{\varphi}_c(0) = \dot{\varphi}(0) \quad .$$

This is accurate when the conservative condition:

$$c(\dot{\varphi}(t)) \ll 1$$

is satisfied. Let $E_\varphi(t) \stackrel{\text{def.}}{=} \frac{1}{2}||\dot{\varphi}(t)||^2 + \Phi(\varphi(t))$ be the cosmological energy of $\varphi$ and set $E_0 = E_\varphi(0)$.

**Proposition**

*We have $||\dot{\varphi}_c(t)|| = \sqrt{2[E_0 - 2\Phi(\varphi_c(t))]}$ and $\mathcal{H}_{\varphi_c} = \frac{1}{M_0}\sqrt{2E_0}$ is independent of t. Moreover, the efold function and IR parameter of $\varphi_c$ are given by:*

$$\mathcal{N}_{\varphi_c}(T) = \frac{1}{3}\int_0^T \mathrm{d}t \mathcal{H}(\dot{\varphi}(t))\mathrm{d}t = \frac{T}{3M_0}\sqrt{2E_0} \quad , \quad \kappa(\dot{\varphi}(t)) = \frac{E_0}{\Phi(\varphi_c(t))} - 1 \quad .$$

*Thus $\dot{\varphi}_c(t)$ is inflationary iff:*

$$\kappa_{\varphi_c}(t) < \frac{1}{2} \Longleftrightarrow \Phi(\varphi_c(t)) > \frac{2E_0}{3} \quad .$$

**Proposition**

*A necessary condition for the conservative approximation to be accurate is* $c_{E_0}(\varphi_c(t)) \ll 1$*, where:*

$$c_{E_0} \stackrel{\mathrm{def.}}{=} \frac{2}{M_0} \frac{[E_0(E_0 - \Phi)]^{1/2}}{||\mathrm{d}\Phi||} = \frac{[E_0(E_0 - \Phi)]^{1/2}}{||\Xi||\Phi} \quad \forall E_0 > 0 \quad .$$

The dissipative approximation considers only non-critical cosmological curves $\varphi$ and neglecting potential term in the cosmological equation, thus replacing $\varphi$ by a solution $\varphi_c$ of the dissipative equation:

$$\nabla_t \dot{\varphi}_d(t) + \mathcal{H}(\varphi_d(t))\dot{\varphi}_d(t) = 0 \quad \text{with} \quad \varphi_d(0) = \varphi(0) \quad \text{and} \quad \dot{\varphi}_d(0) = \dot{\varphi}(0) \quad .$$

This is accurate when the dissipative condition $c_\varphi(t) \gg 1$ is satisfied.

**Proposition**

*The dissipative approximant* $\varphi_d$ *is a reparameterized geodesic of* $(\mathcal{M}, \mathcal{G})$ *whose time and proper legth parameter s are related by the ODE:*

$$t''(s) - \frac{1}{M_0}[||\varphi_d'(s)||^2 + 2\Phi(\varphi_d(s))t'(s)^2]^{1/2} t'(s) = 0$$

*and which satisfies* $\varphi_d(0) = \varphi(0) \quad \text{and} \quad \varphi_d'(0) = \frac{\dot{\varphi}(0)}{||\dot{\varphi}(0)||}$.

# The dissipative approximation

**Proposition**

*A necssary condition for the dissipative approximation to be accurate is:*

$$\frac{\sqrt{1+2t'(s)^2\Phi(\varphi_d(s))}}{2t'(s)^2\Phi(\varphi_d(s))||\Xi(\varphi_d(s))||} = \frac{\sqrt{1+2t'(s)^2\Phi(\varphi_d(s))}}{M_0 t'(s)^2||(\mathrm{d}\Phi)(\varphi_d(s))||} \ll 1$$

Combining the dissipative approximation with the first slow roll approximation produces the UV approximation.

# The small and large Planck mass approximations

When $M_0 \gg 1$, the friction term can be neglected and cosmological curves are approximated by solutions of the conservative equation.

When $M_0 \ll 1$, the scale transformation with parameter $\epsilon = M_0$ brings the cosmological equation to the form:

$$M_0^2 \nabla_t \frac{\mathrm{d}\varphi_{M_0}(t)}{\mathrm{d}t} + \left[M_0^2 || \frac{\mathrm{d}\varphi_{M_0}(t)}{\mathrm{d}t} ||^2 + 2\Phi(\varphi_{M_0}(t))\right]^{1/2} \frac{\mathrm{d}\varphi_{M_0}(t)}{\mathrm{d}t} + (\mathrm{grad}_{\mathcal{G}}\Phi)(\varphi_{M_0}(t)) = 0$$

where $\varphi_{M_0}(t) = \varphi(t/M_0)$. Hence the limit $M_0 \to 0$ coincides with the infrared limit with parameter $\epsilon = M_0$. In this limit, $\varphi(t)$ is well-approximated by the solution $\varphi_0(t)$ of the gradient flow equation of $V$:

$$\frac{\mathrm{d}\varphi_0(t)}{\mathrm{d}t} + (\mathrm{grad} V)(\varphi_0(t)) = 0 \quad \mathrm{with} \quad \varphi_0(0) = \varphi(0) \quad .$$

The approximation is optimal for infrared optimal curves, which satisfy:

$$\dot{\varphi}(0) = -\frac{M_0}{\sqrt{2\Phi}} (\mathrm{grad}\Phi)(\varphi(0)) \quad .$$

The approximation is accurate when:

$$\kappa_\varphi(t) \stackrel{\mathrm{def.}}{=} \frac{||\dot{\varphi}(t)||^2}{2\Phi(\varphi(t))} \ll 1 \quad \mathrm{and} \quad \tilde{\kappa}_\varphi(t) \stackrel{\mathrm{def.}}{=} \frac{||\nabla_t \dot{\varphi}(t)||}{||(\mathrm{d}\Phi)(\varphi(t))||} \ll 1 \quad .$$

One can develop expansions in positive or negative powers of $M_0$.