A new nonholonomic problem: spherical ball bearings

Borislav Gajić
(Join paper with Vladimir Dragović and Božidar Jovanović)

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We consider the following problem: n homogeneous balls $B_1, \ldots, B_n$ with centers $O_1, ..., O_n$ and the same radius r roll without slipping around a fixed sphere S_0 with center O and radius R. A dynamically nonsymmetric sphere S of radius $R + 2r$ with the center that coincides with the center O of the fixed sphere S_0 rolls without slipping over the moving balls $B_1, \ldots, B_n$.

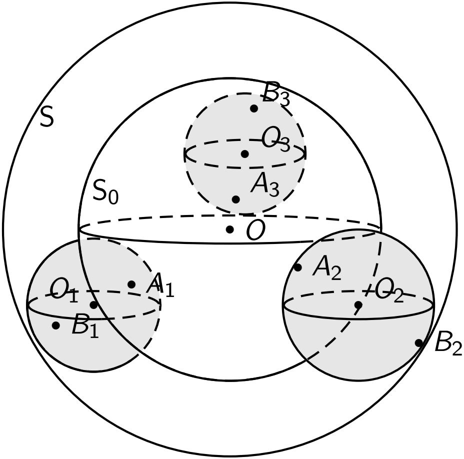

FIGURE 1. Spherical ball bearing for $n = 3$

Let

$$O\vec{e}^0_1, \vec{e}^0_2, \vec{e}^0_3, \qquad O\vec{e}_1, \vec{e}_2, \vec{e}_3, \qquad O_i\vec{e}^i_1, \vec{e}^i_2, \vec{e}^i_3, \qquad i = 1, \dots, n$$

be positively oriented reference frames rigidly attached to the spheres S_0, S, and the balls B_i, $i = 1, \dots, n$, respectively. By $\mathrm{g}, \mathrm{g}_i \in SO(3)$ we denote the matrices that map the moving frames $O\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $O_i\vec{e}^i_1, \vec{e}^i_2, \vec{e}^i_3$ to the fixed frame $O\vec{e}^0_1, \vec{e}^0_2, \vec{e}^0_3$:

$$\mathrm{g}_{jk} = \langle \vec{e}^0_j, \vec{e}_k \rangle, \qquad \mathrm{g}_{i,jk} = \langle \vec{e}^0_j, \vec{e}^i_k \rangle, \qquad j, k = 1, 2, 3, \qquad i = 1, \dots, n.$$

We apply the standard isomorphism between the Lie algebras $(so(3),[\cdot,\cdot])$ and $(\mathbb{R}^3,\times)$

$$a_{ij} = -\varepsilon_{ijk} a_k, \qquad i,j,k = 1,2,3, \tag{0.1}$$

The skew-symmetric matrices

$$\omega = \dot{\mathrm{g}}\mathrm{g}^{-1}, \qquad \omega_i = \dot{\mathrm{g}}_i \mathrm{g}_i^{-1}$$

correspond to the angular velocities $\vec{\omega}$, $\vec{\omega}_i$ of the sphere S and the i-th ball B_i in the fixed reference frame $O\vec{e}^0_1,\vec{e}^0_2,\vec{e}^0_3$ attached to the sphere S_0. The matrices

$$\Omega = \mathrm{g}^{-1}\dot{\mathrm{g}} = \mathrm{g}^{-1}\omega\mathrm{g}, \qquad W_i = \mathrm{g}_i^{-1}\dot{\mathrm{g}}_i = \mathrm{g}_i^{-1}\omega_i\mathrm{g}_i$$

correspond to the angular velocities $\vec{\Omega}$, $\vec{W}_i$ of S and B_i in the frames $O\vec{e}_1,\vec{e}_2,\vec{e}_3$ and $O_i\vec{e}^i_1,\vec{e}^i_2,\vec{e}^i_3$ attached to the sphere S and the balls B_i, respectively.
We have

$$\vec{\omega} = \mathrm{g}\vec{\Omega}, \qquad \vec{\omega}_i = \mathrm{g}_i\vec{W}_i.$$

Let I be the inertia operator of the outer sphere S. We choose the moving frame $O\vec{e}_1, \vec{e}_2, \vec{e}_3$, such that $O\vec{e}_1$, $O\vec{e}_2$, $O\vec{e}_3$ are the principal axes of inertia: $I = \operatorname{diag}(A, B, C)$. Let $\operatorname{diag}(I_i, I_i, I_i)$ and m_i be the inertia operator and the mass of the i-th ball B_i. Then the configuration space and the kinetic energy of the problem are given by:

$$
\begin{aligned}
Q =& SO(3)^{n+1} \times (S^2)^n \{\mathrm{g}, \mathrm{g}_1, \dots, \mathrm{g}_n, \vec{\gamma}_1, \dots, \vec{\gamma}_n\},\\
T =& \frac{1}{2}\langle I\vec{\Omega}, \vec{\Omega}\rangle + \frac{1}{2}\sum_{i=1}^{n} I_i \langle \vec{W}_i, \vec{W}_i\rangle + \frac{1}{2}\sum_{i=1}^{n} m_i \langle \vec{v}_{O_i}, \vec{v}_{O_i}\rangle\\
=& \frac{1}{2}\langle I\vec{\Omega}, \vec{\Omega}\rangle + \frac{1}{2}\sum_{i=1}^{n} I_i \langle \vec{\omega}_i, \vec{\omega}_i\rangle + \frac{1}{2}\sum_{i=1}^{n} m_i \langle \vec{v}_{O_i}, \vec{v}_{O_i}\rangle.
\end{aligned}
$$

Here $\vec{\gamma}_i$ is the unit vector

$$
\vec{\gamma}_i = \frac{\overrightarrow{OO_i}}{|\overrightarrow{OO_i}|}
$$

determining the position O_i of the centre of i-th ball B_i and $\vec{v}_{O_i} = (R + r)\dot{\vec{\gamma}}_i$ is its velocity, $i = 1, \dots, n$.

Let us denote the contact points of the balls $\mathrm{B}_1, \ldots, \mathrm{B}_n$ with the spheres S_0 and S by $A_1, ..., A_n$ and $B_1, B_2, ..., B_n$, respectively. The condition that the rolling of the balls $\mathrm{B}_1, \ldots, \mathrm{B}_n$ and the sphere S are without slipping leads to the nonholonomic constraints:

$$\vec{v}_{O_i} = r\vec{\omega}_i \times \vec{\gamma}_i, \qquad \vec{v}_{O_i} = (R+2r)\vec{\omega} \times \vec{\gamma}_i - r\vec{\omega}_i \times \vec{\gamma}_i, \qquad i = 1, ..., n. \tag{0.2}$$

The dimension of the configuration space Q is $5n+3$. There are $4n$ independent constraints in (0.2), defining a nonintegrable distribution $\mathcal{D} \subset TQ$. Therefore, the dimension of the vector subspaces of admissible velocities $\mathcal{D}_q \subset T_q Q$ is $n+3$, $q \in Q$. The phase space of the system has the dimension $6n+6$, which is the dimension of the bundle $\mathcal{D}$ as a submanifold of TQ.

The kinetic energy and the constraints are invariant with respect to the $SO(3)^{n+1}$–action defined by

$$(\mathrm{g}, \mathrm{g}_1, \dots, \mathrm{g}_n, \vec{\gamma}_1, \dots, \vec{\gamma}_n) \longmapsto (\mathrm{ag}, \mathrm{ag}_1 \mathrm{a}_1^{-1}, \dots, \mathrm{ag}_n \mathrm{a}_n^{-1}, \mathrm{a}\vec{\gamma}_1, \dots, \mathrm{a}\vec{\gamma}_n), \tag{0.3}$$

$\mathrm{a}, \mathrm{a}_1, \dots, \mathrm{a}_n \in SO(3)$, representing a freedom in the choice of the reference frames

$$O\vec{\mathrm{e}}_1^0, \vec{\mathrm{e}}_2^0, \vec{\mathrm{e}}_3^0, \qquad O_i\vec{\mathrm{e}}_1^i, \vec{\mathrm{e}}_2^i, \vec{\mathrm{e}}_3^i, \qquad i = 1, \dots, n.$$

Thus, for the coordinates in the space $(TQ)/SO(3)^{n+1}$ we can take the angular velocities and the unit position vectors in the reference frame attached to the sphere S:

$$(TQ)/SO(3)^{n+1} \cong \mathbb{R}^{3(n+1)} \times (TS^2)^n \{\vec{\Omega}, \vec{\Omega}_1, \dots, \vec{\Omega}_n, \dot{\vec{\Gamma}}_1, \dots, \dot{\vec{\Gamma}}_n, \vec{\Gamma}_1, \dots, \vec{\Gamma}_n\}.$$

In the moving reference frame $O\vec{e}_1, \vec{e}_2, \vec{e}_3$, the constraints become:

$$\vec{V}_{O_i} = (R + 2r)\vec{\Omega} \times \vec{\Gamma}_i - r\vec{\Omega}_i \times \vec{\Gamma}_i, \tag{0.4}$$

$$\vec{V}_{O_i} = r\vec{\Omega}_i \times \vec{\Gamma}_i, \qquad i = 1, \dots, n, \tag{0.5}$$

defining the *reduced phase space*
$\mathcal{M} = \mathcal{D}/SO(3)^{n+1} \subset (TQ)/SO(3)^{n+1}$ of dimension $3n + 3$.
Since both the kinetic energy and the constraints are invariant with respect to the $SO(3)^{n+1}$–action (0.3), the equations of motion are also $SO(3)^{n+1}$–invariant. Thus, they induce a well defined system on the reduced phase space $\mathcal{M}$.

Lemma

The kinematic part of the equations of motion of the spherical ball bearing system is:

$$\dot{\vec{\Gamma}}_i = \frac{R}{2R+2r}\vec{\Gamma}_i \times \vec{\Omega}, \qquad i = 1, \ldots, n. \tag{0.6}$$

Proof. Consider the fixed reference frame $O\vec{e}_1^0, \vec{e}_2^0, \vec{e}_3^0$. One has $\dot{\overrightarrow{OO}}_i + \vec{\omega}_i \times \overrightarrow{O_iA_i} = 0$.
Therefore, $(R+r)\dot{\vec{\gamma}}_i - r\vec{\omega}_i \times \vec{\gamma}_i = 0$, or equivalently

$$\dot{\vec{\gamma}}_i = \frac{r}{R+r}\vec{\omega}_i \times \vec{\gamma}_i.$$

The equation in the moving reference frame $O\vec{e}_1, \vec{e}_2, \vec{e}_3$ has the form

$$\dot{\vec{\Gamma}}_i + \vec{\Omega} \times \vec{\Gamma}_i = \frac{r}{R+r}\vec{\Omega}_i \times \vec{\Gamma}_i.$$

Thus, we get

$$\dot{\vec{\Gamma}}_i = (\frac{r}{R+r}\vec{\Omega}_i - \vec{\Omega}) \times \vec{\Gamma}_i.$$

From the constraints we obtain

$$\vec{\Omega}_i \times \vec{\Gamma}_i = \frac{R+2r}{2r}\vec{\Omega} \times \vec{\Gamma}_i, \qquad i = 1, \dots, n.$$

Finally, the equations can be written in a more convenient form

$$\dot{\vec{\Gamma}}_i = \frac{R}{2R+2r}\vec{\Gamma}_i \times \vec{\Omega}, \qquad i = 1, \dots, n.$$

As a consequence, we have:

Proposition

The following functions are the first integrals of motion:

$$\langle \vec{\Gamma}_i, \vec{\Gamma}_j \rangle = \gamma_{ij} = const, \qquad i, j = 1, \dots, n.$$

In other words, the centers O_i of the homogeneous balls B_i are in rest in relation to each other.

Let $\vec{\mathsf{F}}_{B_i}$ and $\vec{\mathsf{F}}_{A_i}$ be the reaction forces that act on the ball B_i at the points B_i and A_i, respectively. The reaction force at the point B_i on the sphere S is then $-\vec{\mathsf{F}}_{B_i}$.

Lemma

The dynamical part of the equations of motion of the spherical ball bearing system is:

$$
\begin{aligned}
I_i\dot{\vec{\Omega}}_i &= I_i\vec{\Omega}_i \times \vec{\Omega} + r\vec{\Gamma}_i \times (\vec{\mathsf{F}}_{B_i} - \vec{\mathsf{F}}_{A_i}), \\
m_i\dot{\vec{V}}_{O_i} &= m_i\vec{V}_{O_i} \times \vec{\Omega} + \vec{\mathsf{F}}_{B_i} + \vec{\mathsf{F}}_{A_i}, \qquad\qquad i = 1, ..., n \\
I\dot{\vec{\Omega}} &= I\vec{\Omega} \times \vec{\Omega} - \sum_{i=1}^{n}(R+2r)\vec{\Gamma}_i \times \vec{\mathsf{F}}_{B_i}.
\end{aligned}
$$

Proposition

The projections of the angular velocities $\vec{\Omega}_i$ to to the directions $\vec{\Gamma}_i$ are the first integrals of motion:

$$\langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle = c_i = const, \qquad i = 1, ..., n.$$

Proof.

$$\begin{aligned} \frac{d}{dt}\langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle =& \langle \dot{\vec{\Omega}}_i, \vec{\Gamma}_i \rangle + \langle \vec{\Omega}_i, \dot{\vec{\Gamma}}_i \rangle \\ =& \langle \vec{\Omega}_i \times \vec{\Omega}, \vec{\Gamma}_i \rangle + \langle \frac{r}{I_i} \vec{\Gamma}_i \times (\vec{F}_{B_i} - \vec{F}_{A_i}), \vec{\Gamma}_i \rangle \\ & + \langle \vec{\Omega}_i, \frac{r}{R+r} \vec{\Omega}_i \times \vec{\Gamma}_i \rangle - \langle \vec{\Omega}_i, \vec{\Omega} \times \vec{\Gamma}_i \rangle = 0. \end{aligned}$$

The reduced system

From the constraints we get

$$\langle \vec{\Omega} \times \vec{\Gamma}_i, \vec{\Omega}_i \rangle = 0.$$

Moreover, we obtain:

$$\vec{\Omega}_i = \langle \vec{\Gamma}_i, \vec{\Omega}_i \rangle \vec{\Gamma}_i + \frac{R+2r}{2r}\vec{\Omega} - \frac{R+2r}{2r}\langle \vec{\Gamma}_i, \vec{\Omega} \rangle \vec{\Gamma}_i.$$

Further, we get that the reduced phase space $\mathcal{M} = \mathcal{D}/SO(3)^{n+1}$ is foliated on $2n+3$–dimensional invariant varieties

$$\mathcal{M}_c : \qquad \langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle = c_i = const, \qquad i = 1, ..., n.$$

On the invariant variety $\mathcal{M}_c$, the vector-functions $\vec{\Omega}_i$ can be uniquely expressed as functions of $\vec{\Omega}$, $\vec{\Gamma}_i$:

$$\vec{\Omega}_i = c_i \vec{\Gamma}_i + \frac{R+2r}{2r}\vec{\Omega} - \frac{R+2r}{2r}\langle \vec{\Gamma}_i, \vec{\Omega} \rangle \vec{\Gamma}_i.$$

Whence, $\vec{\Omega}$ determines all velocities of the system on $\mathcal{M}_c$ and $\mathcal{M}_c$ is diffeomorphic to the *second reduced phase space*

$$\mathcal{N} = \mathbb{R}^3 \times \left(S^2\right)^n \{\vec{\Omega}, \vec{\Gamma}_1, \dots, \vec{\Gamma}_n\}.$$

Thus, instead of the derivation of the torques of all reaction forces, it is sufficient to find the torque in the equation on a given invariant variety $\mathcal{M}_c$. To simplify the equations, we introduce the parameters

$$\varepsilon = \frac{R}{2R+2r} \qquad \text{and} \qquad \delta = \frac{R+2r}{2r}.$$

We define the *modified operator of inertia* I as

$$\mathsf{I} = I + \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2)\,\mathrm{pr}_i,$$

where $\mathrm{pr}_i \colon \mathbb{R}^3 \to \vec{\Gamma}_i^{\perp}$ is the orthogonal projection to the plane orthogonal to $\vec{\Gamma}_i$. We set

$$\vec{M} = \mathsf{I}\vec{\Omega} = I\vec{\Omega} + \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2)\vec{\Omega} - \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2)\langle \vec{\Gamma}_i, \vec{\Omega}\rangle \vec{\Gamma}_i,$$

$$\vec{N} = \delta \sum_{i=1}^{n} I_i c_i \vec{\Gamma}_i.$$

Theorem

The reduction of the spherical ball bearing problem to $\mathcal{M}_c \cong \mathcal{N}$ is described by the equations

$$\frac{d}{dt}\vec{M} = \vec{M} \times \vec{\Omega} + (1-\varepsilon)\vec{N} \times \vec{\Omega}, \tag{0.7}$$

$$\frac{d}{dt}\vec{\Gamma}_i = \varepsilon\vec{\Gamma}_i \times \vec{\Omega}, \qquad i = 1, \dots, n. \tag{0.8}$$

The kinetic energy of the system takes the form

$$T = \frac{1}{2}\langle \vec{M}, \vec{\Omega} \rangle + \frac{1}{2}\sum_{i=1}^{n} I_i c_i^2.$$

Also, since

$$\frac{d}{dt}\vec{N} = \varepsilon\vec{N} \times \vec{\Omega},$$

the equation (0.7) is equivalent to

$$\frac{d}{dt}(\vec{M} + \vec{N}) = (\vec{M} + \vec{N}) \times \vec{\Omega}. \tag{0.9}$$

Proof. From the equations one have

$$\vec{\Gamma}_i \times \vec{F}_{B_i} = \frac{1}{2r}(I_i\dot{\vec{\Omega}}_i + \vec{\Omega} \times (I_i\vec{\Omega}_i)) + \frac{m_i}{2}\vec{\Gamma}_i \times \dot{\vec{V}}_{O_i} + \frac{m_i}{2}\vec{\Gamma}_i \times (\vec{\Omega} \times \vec{V}_{O_i})$$

By plugging the last expression in the third equation of motion, it becomes

$$\begin{aligned} I\dot{\vec{\Omega}} + \vec{\Omega} \times I\vec{\Omega} = -\sum_{i=1}^{n} \Big[& \frac{R+2r}{2r}(I_i\dot{\vec{\Omega}}_i + \vec{\Omega} \times (I_i\vec{\Omega}_i)) + \\ & \frac{m_i(R+2r)}{2}\vec{\Gamma}_i \times \dot{\vec{V}}_{O_i} + \frac{m_i(R+2r)}{2}\vec{\Gamma}_i \times (\vec{\Omega} \times \vec{V}_{O_i})\Big]. \end{aligned} \tag{0.10}$$

We get $\dot{\vec{\Gamma}}_i \times \vec{V}_{O_i} = 0$, and, therefore

$$\frac{d}{dt}\big(\vec{\Gamma}_i \times \vec{V}_{O_i}\big) = \vec{\Gamma}_i \times \dot{\vec{V}}_{O_i}.$$

Also, we have

$$\vec{\Gamma}_i \times (\vec{\Omega} \times \vec{V}_{O_i}) = \vec{\Omega} \times (\vec{\Gamma}_i \times \vec{V}_{O_i}).$$

Having in mind the last two expressions, the equation (0.10) becomes

$$\frac{d}{dt}\Big(I\vec{\Omega} + \sum_{i=1}^{n} \big(\frac{R+2r}{2r} I_i \vec{\Omega}_i + \frac{m_i(R+2r)}{2}\vec{\Gamma}_i \times \vec{V}_{O_i}\big)\Big) = -\vec{\Omega} \times \Big(I\vec{\Omega} + \sum_{i=1}^{n} \big(\frac{R+2r}{2r} I_i \vec{\Omega}_i + \frac{m_i(R+2r)}{2}\vec{\Gamma}_i \times \vec{V}_{O_i}\big)\Big) \tag{0.11}$$

Finally, using the definitions of parameters ε and δ and the vectors $\vec{M}$ and $\vec{N}$, the equation (0.11) takes the form (0.9).

Remark

If we formally set $\varepsilon = 1$ in the system, we obtain the equation of the spherical support system introduced by Fedorov (Vestnik MGU 1988). The system describes the rolling without slipping of a dynamically nonsymmetric sphere S over n homogeneous balls $B_1, \dots, B_n$ of possibly different radii, but with fixed centers. It is an example of a class of nonhamiltonian L+R systems on Lie groups with an invariant measure. On the other hand, if we set $\vec{N} = 0$, we obtain an example ε–modified L+R system studied by Jovanovic (RCD 2015).
The rolling of a homogeneous ball over a dynamically asymmetric sphere S is introduced by Borisov, Kilin, and Mamaev (RCD, 2011)

Corollary

The complete equations of motion of the sphere S *and the balls* $\mathrm{B}_1, \ldots, \mathrm{B}_n$ *of the spherical ball bearing problem on the invariant manifold* $\mathcal{D}_c$ *are given by*

$$
\begin{aligned}
\dot{\vec{M}} &= \vec{M} \times \vec{\Omega} + (1-\varepsilon)\vec{N} \times \vec{\Omega}, \\
\dot{\mathrm{g}} &= \mathrm{g}\Omega, \\
\dot{\mathrm{g}}_i &= \mathrm{g}\Omega_i(\vec{\Omega}, \vec{\Gamma}_i, c_i)\mathrm{g}_i, \\
\dot{\vec{\Gamma}}_i &= \varepsilon\vec{\Gamma}_i \times \vec{\Omega}, \qquad\qquad i = 1, \ldots, n,
\end{aligned}
$$

Here Ω *and* $\Omega_i(\vec{\Omega}, \vec{\Gamma}_i, c_i)$ *are skew-symmetric matrices related to* $\vec{\Omega}$ *and* $\vec{\Omega}_i$*;* $\vec{\Omega}_i = \vec{\Omega}_i(\vec{\Omega}, \vec{\Gamma}_i, c_i)$*.*

The associated system on $\mathbb{R}^3 \times Sym(3)$ and an invariant measure

Let

$$\Gamma = -\delta^2 \sum_{i=1}^{n} (I_i + m_i r^2)\,\mathrm{pr}_i$$

be the symmetric operator. Then the modified inertia operator I can be rewritten as:

$$\mathrm{I} = I - \Gamma, \qquad \Gamma = \delta^2 \sum_{i=1}^{n} (I_i + m_i r^2)\big(\vec{\Gamma}_i \otimes \vec{\Gamma}_i - \mathrm{E}\big), \qquad \mathrm{E} = \mathrm{diag}(1, 1, 1).$$

Along the flow of the system, Γ satisfies the equation

$$\frac{d}{dt}\Gamma = \varepsilon[\Gamma, \Omega], \tag{0.12}$$

where Ω is the skew-symmetric matrix that corresponds to $\vec{\Omega}$.
Let us consider a special case when $c_1 = 0, \dots, c_n = 0$. This means that there are no twisting of the balls, i.e. the vectors $\vec{\Omega}_i$ and $\vec{\Gamma}_i$ are orthogonal to each other. Note that this conditions are not nonholonomic constraints, but the first integrals of motion.

As a result we obtain the associated system

$$\dot{\vec{M}} = \vec{M} \times \vec{\Omega}, \qquad \vec{M} = \mathsf{I}\vec{\Omega} = I\vec{\Omega} - \Gamma\vec{\Omega}, \\ \dot{\Gamma} = \varepsilon[\Gamma, \Omega] \tag{0.13}$$

on the space $\mathbb{R}^3 \times Sym(3)$, where $Sym(3)$ are 3×3 symmetric matrices. The system belongs to the class of ε–modified L+R systems studied by Jovanovic (RCD, 2015).
Let $d\Omega$ and $d\Gamma$ be the standard measures on $\mathbb{R}^3\{\vec{\Omega}\}$ and $Sym(3)\{\Gamma\}$. The system (0.13) possesses the invariant measure $\mu(\Gamma)d\Omega \wedge d\Gamma$ with the density $\mu(\Gamma) = \sqrt{\det(\mathsf{I})}$ Jovanovic (RCD, 2015) Therefore, $\mu = \sqrt{\det(\mathsf{I})}$ is a natural candidate for the density of an invariant measure of the system (0.7), (0.8) when the constants c_i are different from zero.

Theorem

For arbitrary values of parameters c_i, the reduced system (0.7), (0.8) *has the invariant measure*

$$\mu(\vec{\Gamma}_1, \dots, \vec{\Gamma}_n) d\Omega \wedge \sigma_1 \wedge \cdots \wedge \sigma_n, \qquad \mu = \sqrt{\det(\mathrm{I})} = \sqrt{\det(I - \Gamma)}, \tag{0.14}$$

where $d\Omega$ and σ_i are the standard measures on $\mathbb{R}^3\{\vec{\Omega}\}$ and $S^2\{\vec{\Gamma}_i\}$, $i = 1, \dots, n$.

The proof of the Theorem is a variant of a corresponding proof for ε–modified L+R systems (Jovanovic, RCD 2015). It uses

Lemma

Let A be a symmetric matrix and let $\vec{\Omega} \in \mathbb{R}^3$ corresponds to $\Omega \in so(3)$. Then:

(i) *the symmetric part of the matrix $\partial\big(A\vec{\Omega} \times \vec{\Omega}\big)/\partial\vec{\Omega}$ is equal to $\frac{1}{2}[A, \Omega]$;*

(ii) $A\vec{\Omega} \times \vec{\Omega} = [A, \Omega]\vec{\Omega}$.

Note that the existence of an invariant measure for nonholonomic problems is well studied. A closely related problem is the integrability of the nonholonomic systems. Here we have the following statement.

Proposition

The system (0.7), (0.8) *always has the following first integrals*

$$F_1 = \frac{1}{2}\langle \vec{M}, \vec{\Omega} \rangle, \quad F_2 = \langle \vec{M} + \vec{N}, \vec{M} + \vec{N} \rangle, \quad F_{ij} = \langle \vec{\Gamma}_i, \vec{\Gamma}_j \rangle, \quad 1 \leq i < j \leq n.$$

Thus, in the special case $n = 1$, we have the 5-dimensional phase space $\mathcal{N} = \mathbb{R}^3 \times S^2\{\vec{\Omega}, \vec{\Gamma}_1\}$, and the system has two first integrals and an invariant measure. For the integrability, one needs to find a third independent first integral. We will study integrability in the spherical ball bearing problems in a separate paper. Also, it would be interesting to study the appropriate nonholonomic systems in arbitrary dimension $\mathbb{R}^m$, $m > 3$.

Planar system - the three balls bearings problem

Consider the limit, when the radii of the spheres S_0 and S both tend to infinity. For simplicity, we consider the case $n = 3$. As a result, we obtain rolling without slipping of three homogeneous balls $\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3$ of the radius r and masses m_1, m_2, m_3 over the fixed plane Σ_0, together with the moving plane Σ of the mass m that is placed over the balls, such that there is no slipping between the balls and moving plane. We will refer to the system as *the planar three balls bearing problem*. All considerations can be easily adopted for the case of the planar ball bearing with rolling of n homogeneous balls.

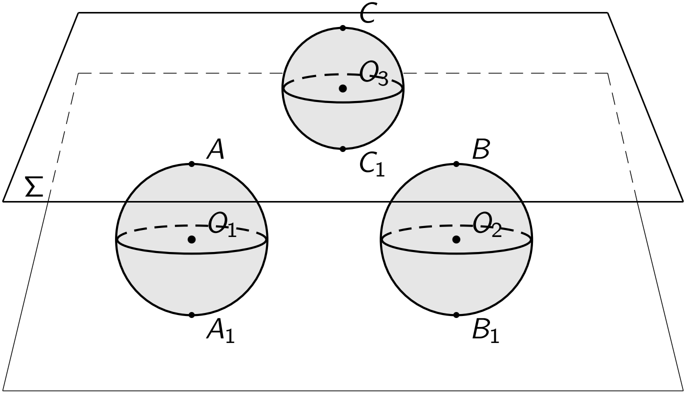

FIGURE 2. Planar three balls bearing problem

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