Entanglement of Gaussian states in curved spacetime

Madalin Calamanciuc and Aurelian Isar

Department of Theoretical Physics
National Institute of Physics and Nuclear Engineering
Bucharest-Magurele, Romania
isar@theory.nipne.ro

SEENET-MTP BWAM22
Belgrade, Serbia
1-4 September 2022

Outline

- We study the Hawking effect on quantum entanglement for two-mode Gaussian states in the presence of a Schwarzschild black hole.
- For a two-mode bosonic system, starting with an initial squeezed thermal state, the influence of Hawking radiation consists in destroying the entanglement between the mode observed by an inertial observer Alice and a mode described by an accelerated observer Bob that hovers near the event horizon of the black hole.
- However in the case of the modes described by Bob and an imaginary observer inside the event horizon anti-Bob quantum entanglement is created, i.e. entanglement is created for causally disconnected modes.

Introduction

- Realistic q. ss are non-inertial and they manifest relativistic and gravitational characteristics $\rightarrow$ relativistic q. investigations are of basic importance, for applications in QI protocols, but also for a better understanding of the features of our universe.
- It has been intensively investigated the evolution of q. correlations in relativistic setting, QI processing tasks using q. correlations; recent studies have shown that entanglement is an important ingredient in the physics of black holes (BH).
- We describe q. field dynamics for free massless bosonic modes in background of a Schwarzschild BH. S.: stationary observer Alice at an asymptotically flat region (or falls freely into BH), with associated mode A, observer Bob who hovers near event horizon of BH with uniform acceleration, associated with mode B, and imaginary observer anti-Bob inside event horizon, associated with mode $\bar{B}$. We assume that Alice and Bob share a Gaussian two-mode STS and study the effect of the Schwarzschild BH on q. entanglement in the s., by considering different scenarios.

Unruh-Hawking effect

- The radiation of the black hole due to the Unruh-Hawking effect can be described by a Gaussian bosonic amplification channel.
- The metric characterising the spacetime background near a static and asymptotically flat Schwarzschild black hole has the following form (we employ the natural units by setting $\hbar = G = c = k_B = 1$):

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\ &+ r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \end{aligned} \tag{1}$$

where M is the mass of the black hole.

Unruh-Hawking effect

- A massless bosonic field ϕ in the background of the black hole satisfies the following Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\phi}{\partial x^{\nu}}\right)=0, \tag{2}$$

  The bosonic field at the black hole can be divided into two regions: inside and outside of the event horizon. By solving the Klein-Gordon equation near the event horizon of the Schwarzschild black hole one obtains the following positive-frequency outgoing associated modes inside and outside of the event horizon:

$$\begin{aligned}\Phi^{+}_{\Omega,\text{in}} &\sim \phi(r)e^{\mathrm{i}\omega u},\\ \Phi^{+}_{\Omega,\mathrm{out}} &\sim \phi(r)e^{-\mathrm{i}\omega u},\end{aligned} \tag{3}$$

  where $u=t-(r+2M\ln\frac{r-2M}{2M})$.

Unruh-Hawking effect

- Using the Schwarzschild modes (3) one can expand the scalar field near the event horizon as follows:

$$\phi = \int d\Omega \left[\hat{a}_{\Omega}^{\mathrm{out}} \Phi^{+}_{\Omega,\mathrm{out}} + \hat{b}_{\Omega}^{\mathrm{out}\dagger} \Phi^{-}_{\Omega,\mathrm{out}} + \hat{a}_{\Omega}^{\mathrm{in}} \Phi^{+}_{\Omega,\mathrm{in}} + \hat{b}_{\Omega}^{\mathrm{in}\dagger} \Phi^{-}_{\Omega,\mathrm{in}} \right]. \tag{4}$$

  Here $\hat{a}_{\Omega}^{\mathrm{out}}$ and $\hat{b}_{\Omega}^{\mathrm{out}\dagger}$ denote the boson annihilation and antiboson creation operators acting on the state outside of the black hole, respectively, and $\widehat{a}_{\Omega}^{\mathrm{in}}$ and $\widehat{b}_{\Omega}^{\mathrm{in}\dagger}$ are the boson annihilation and antiboson creation operators acting on the inside states, respectively.

Unruh-Hawking effect

- In order to find the relation between the bosonic field in the flat spacetime and the bosonic field at the black hole, one introduces the Unruh operators, which are related to the Schwarzschild operators through the following Bogoliubov transformations:

$$
\begin{aligned}
C_{\Omega,\mathrm{R}} &= \left(\cosh r_\Omega\ \hat{a}_{\Omega,\mathrm{out}} - \sinh r_\Omega\ \hat{b}^\dagger_{\Omega,\mathrm{in}}\right), \\
C_{\Omega,\mathrm{L}} &= \left(\cosh r_\Omega\ \hat{a}_{\Omega,\mathrm{in}} - \sinh r_\Omega\ \hat{b}^\dagger_{\Omega,\mathrm{out}}\right), \\
D^\dagger_{\Omega,\mathrm{R}} &= \left(-\sinh r_\Omega\ \hat{a}_{\Omega,\mathrm{out}} + \cosh r_\Omega\ \hat{b}^\dagger_{\Omega,\mathrm{in}}\right), \\
D^\dagger_{\Omega,\mathrm{L}} &= \left(-\sinh r_\Omega\ \hat{a}_{\Omega,\mathrm{in}} + \cosh r_\Omega\ \hat{b}^\dagger_{\Omega,\mathrm{out}}\right),
\end{aligned}
\tag{5}
$$

where $\sinh r_\Omega = \left(e^{\frac{\Omega}{T_H}} - 1\right)^{-\frac{1}{2}}$ and T_H is the Hawking temperature of the black hole. The Hawking temperature parameter r_Ω is a monotonically increasing function of T_H.

- By introducing a generic Schwarzschild-Fock state $|nm, pq\rangle_\Omega$, describing the particles and antiparticles of the event horizon, the Unruh vacuum can be written as

$$|0_\Omega\rangle_{\mathrm{U}} = \frac{1}{\cosh^2 r_\Omega} \sum_{n,m=0}^{\infty} (\tanh r_\Omega)^{n+m} \, |nn, mm\rangle_\Omega , \tag{6}$$

each Unruh mode Ω being mapped into a Schwarzschild mode Ω.

Unruh-Hawking effect

- Let us consider a bipartite system, where Alice stays stationary at the asymptotically flat region, and Bob is a Schwarzschild observer who hovers with a uniform acceleration near the event horizon of the black hole. The vacuum state in the single-mode approximation, when only bosons are living outside (i.e. only particles can be detected as Hawking radiation) and antibosons are living inside the regions of the event horizon, becomes (we abreviate $r_\Omega \equiv r$ for simplicity)

$$|0_\Omega\rangle_{\mathrm{H}} = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (\tanh r)^n |n\rangle_{\mathrm{out}} \, |n\rangle_{\mathrm{in}} \, , \tag{7}$$

where $|n\rangle_{\mathrm{out}}$ and $|n\rangle_{\mathrm{in}}$ denote the bosonic and antibosonic states, outside and inside of the event horizon, which are observed by the observer Bob and, respectively, by the imaginary observer anti-Bob.

Unruh-Hawking effect

- The right-hand side of Eq. (7) can be written as the action of the two-mode squeezing operator $\hat{U}(r)$ on the associated vacuum states $|0\rangle_{\text{in}}$ and $|0\rangle_{\text{out}}$, inside and outside of the event horizon, respectively. The two-mode squeezing operator is defined as

$$\hat{U}(r) = e^{r\left(\hat{a}^{\dagger}_{\Omega,\text{out}}\hat{b}^{\dagger}_{\Omega,\text{in}} - \hat{a}_{\Omega,\text{out}}\,\hat{b}_{\Omega,\text{in}}\right)}. \tag{8}$$

Then Eq. (7) implies that the Unruh-Hawking radiation of the black hole can be described by a bosonic amplification channel with squeezing operator $\hat{U}(r)$. The squeezing transformation $\hat{U}(r)$ is a Gaussian operation, which preserves the Gaussian form of the input states. The symplectic phase-space representation of $\hat{U}(r)$:

$$S_{B,\bar{B}}(r) = \begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix}. \tag{9}$$

- We consider the massless scalar field ϕ for two Unruh modes A of Alice and B of Bob, who in the inertial frame share a two-mode Gaussian state ρ_{AB}. We denote by $\mathbf{R} = \{x, p_x, y, p_y\}^{\mathrm{T}}$ the vector of canonically conjugated quadrature operators for the two bosonic modes and by σ_{AB} the 4×4 bimodal covariance matrix, with the elements given by the second statistical moments of the quadrature operators

$$\sigma_{ij} = \mathrm{Tr}[(R_i R_j + R_j R_i)\rho_{AB}], i, j = 1, \ldots, 4, \qquad (10)$$

which fully characterise any Gaussian state of a bimodal system. We neglect the first moments, since they can be made zero by suitable local displacements in the phase space. The phase-space operators R_i $(i = 1, .., 4)$ satisfy the canonical commutation relations $[R_i, R_j] = \mathrm{i}\Omega_{ij}$ and the covariance matrix satisfies the uncertainty relation $\sigma_{AB} + \mathrm{i}\Omega_{AB} \geq 0$, where $\Omega_{AB} = \oplus_{i=1}^{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the symplectic form.

Unruh-Hawking effect

The change from Unruh modes to Schwarzschild modes is given by the previously introduced amplification channel, which corresponds to the two-mode squeezing operation associated with the symplectic transformation (9). After this amplification, mode B is mapped into two modes, B and $\bar{B}$ situated outside and, respectively, inside of the event horizon. Therefore, although from an inertial point of view the system is bipartite, from the perspective of a Schwarzschild observer an additional mode $\bar{B}$ becomes relevant. Consequently, the initial bipartite state is mapped into a state of three modes: mode A of Alice, mode B described by Schwarzschild observer Bob, and mode $\bar{B}$, described by a hypothetical observer anti-Bob, situated inside the event horizon of the Scharzschild black hole, and the description of the complete system will involve all three modes associated with the observers with the CM

$$\sigma_{AB\bar{B}}(s,r) = \left[I_A \oplus S_{B,\bar{B}}(r)\right]\left[\sigma^0_{AB}(s) \oplus I_{\bar{B}}\right]\left[I_A \oplus S_{B,\bar{B}}(r)\right]^{\mathrm{T}}. \tag{11}$$

Quantifying Gaussian quantum entanglement

- In order to describe the quantum correlations between the modes of the considered system, we suppose that Alice and Bob share a two-mode Gaussian state of bosonic fields, namely we assume that the state shared by Alice and Bob is a two-mode squeezed thermal state with the following covariance matrix in the inertial frame:

$$\sigma^0_{AB} = \begin{pmatrix} \mathfrak{a}_0 & 0 & \mathfrak{c}_0 & 0 \\ 0 & \mathfrak{a}_0 & 0 & -\mathfrak{c}_0 \\ \mathfrak{c}_0 & 0 & \mathfrak{b}_0 & 0 \\ 0 & -\mathfrak{c}_0 & 0 & \mathfrak{b}_0 \end{pmatrix}, \tag{12}$$

$$\begin{aligned} \mathfrak{a}_0 &= 2n_1 \cosh^2 s + 2n_2 \sinh^2 s + \cosh 2s, \\ \mathfrak{b}_0 &= 2n_2 \cosh^2 s + 2n_1 \sinh^2 s + \cosh 2s, \\ \mathfrak{c}_0 &= (n_1 + n_2 + 1) \sinh 2s. \end{aligned} \tag{13}$$

Here s is the squeezing parameter of the initial state and n_1, n_2 are the associated average thermal photon numbers.

Quantifying Gaussian quantum entanglement

- An observer living outside the region of the BH is causally disconnected from the inside region, therefore Alice, who stays in the asymptotically flat region, and Bob, who hovers with a uniform acceleration near the horizon of the Schwarzschild BH, cannot access the mode $\bar{B}$. In order to obtain the covariance matrix for the outside region we have to perform the trace over the mode that lives inside the BH, associated with the imaginary observer anti-Bob. By performing the trace over $\bar{B}$ in the relation (11), we obtain the covariance matrix of Alice and Bob:

$$\sigma_{AB}(s,r) = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C}^{\mathrm{T}} & \mathcal{B} \end{pmatrix}, \tag{14}$$

$$\begin{aligned} \mathcal{A} &= \mathfrak{a}_0 I, \\ \mathcal{B} &= \left[\mathfrak{b}_0 \cosh^2 r + \sinh^2 r\right] I, \\ \mathcal{C} &= \mathfrak{c}_0 \cosh r \, Z, \end{aligned} \tag{15}$$

I is the identity matrix and Z is the Z-Pauli matrix.

Quantifying Gaussian quantum entanglement

- We now proceed to quantify the quantum entanglement using as a measure the logarithmic negativity, defined in terms of the symplectic invariants of the covariance matrix:

$$E_N = -\log_2 g(\sigma), \tag{16}$$

$$g(\sigma) = \frac{1}{\sqrt{2}} \times$$

$$\sqrt{\det \mathcal{A} + \det \mathcal{B} - 2\det \mathcal{C} - \sqrt{(\det \mathcal{A} + \det \mathcal{B} - 2\det \mathcal{C})^2 - 4\det \sigma}}$$

For $E_N \leq 0$ the state is separable and $E_N > 0$ determines the strength of the entanglement.
By setting the average thermal photon numbers to 0, then the state becomes a squeezed vacuum state, which is entangled for non-zero squeezing.

Results

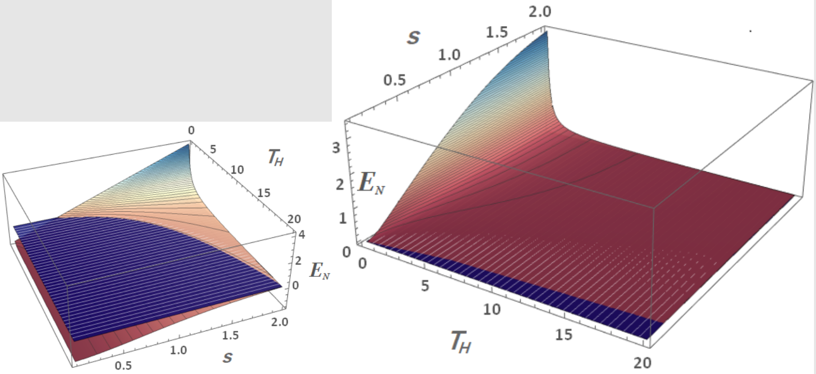

Figure: Alice-Bob Gaussian quantum entanglement

Results

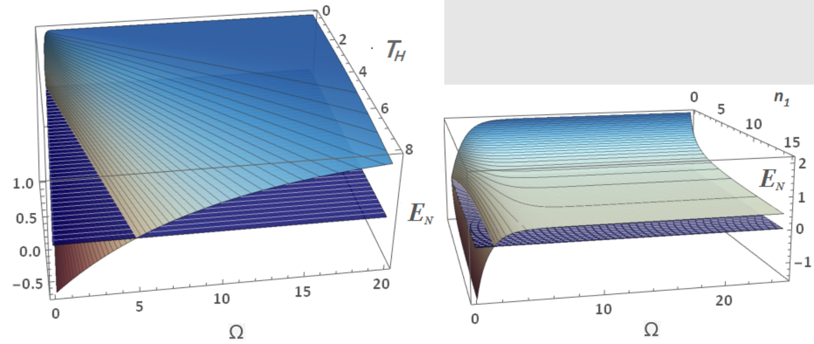

Figure: Alice-Bob Gaussian quantum entanglement

Quantifying Gaussian quantum entanglement

- Recent studies have shown that interesting results are obtained by studying quantum correlations between causally disconnected regions of spacetime. The study of quantum entanglement between physically inaccessible regions could facilitate a better understanding of the connection between quantum information and black hole physics. In order to obtain the covariance matrix for Bob and anti-Bob one has to perform in Eq. (11) the partial trace over the mode observed by Alice. Therefore, for the causally disconnected regions we obtain:

$$\begin{aligned}
\mathcal{A} &= \left[\mathfrak{b}_0 \cosh^2 r + \sinh^2 r\right] I, \\
\mathcal{B} &= \left[\cosh^2 r + \mathfrak{b}_0 \sinh^2 r\right] I, \\
\mathcal{C}_{B\bar{B}} &= \left[(1 + \mathfrak{b}_0) \sinh r \cosh r\right] Z.
\end{aligned} \tag{17}$$

Results

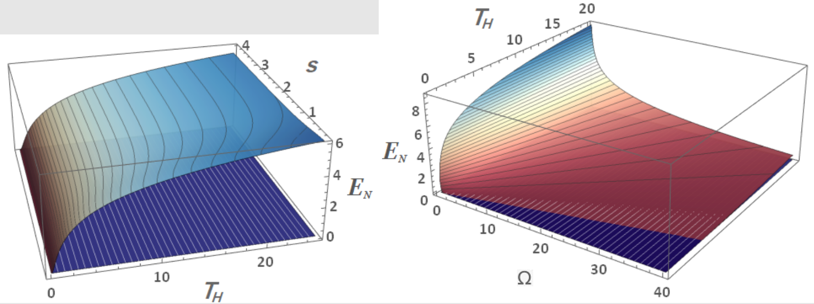

Figure: Bob-anti-Bob Gaussian quantum entanglement

Results

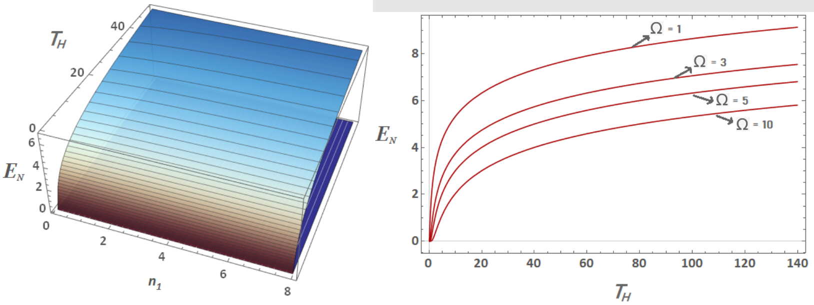

Figure: Bob-anti-Bob Gaussian quantum entanglement

Quantifying Gaussian quantum entanglement

- Finally, we analyse the existence of quantum correlations between Alice and anti-Bob modes. In this case their covariance matrix can be obtained by performing in Eq. (11) the partial trace over the mode observed by Bob:

$$\begin{aligned} \mathcal{A} &= \mathfrak{a}_0 I, \\ \mathcal{B} &= \left[\cosh^2 r + \mathfrak{b}_0 \sinh^2 r\right] I, \\ \mathcal{C}_{A\bar{B}} &= \mathfrak{c}_0 \sinh r Z, \end{aligned} \tag{18}$$

  and the calculations show that $E_N < 0$, so that the state of Alice and anti-Bob is always separable. We illustrate the evolution of the logarithmic negativity, as a function of the thermal photon number n_1, for the modes associated with all of the observers: Alice and Bob (yellow), Alice and anti-Bob (red), Bob and anti-Bob (blue). We can notice again that quantum entanglement for Alice-Bob and Bob-anti-Bob is decreasing by increasing n_1.

Results

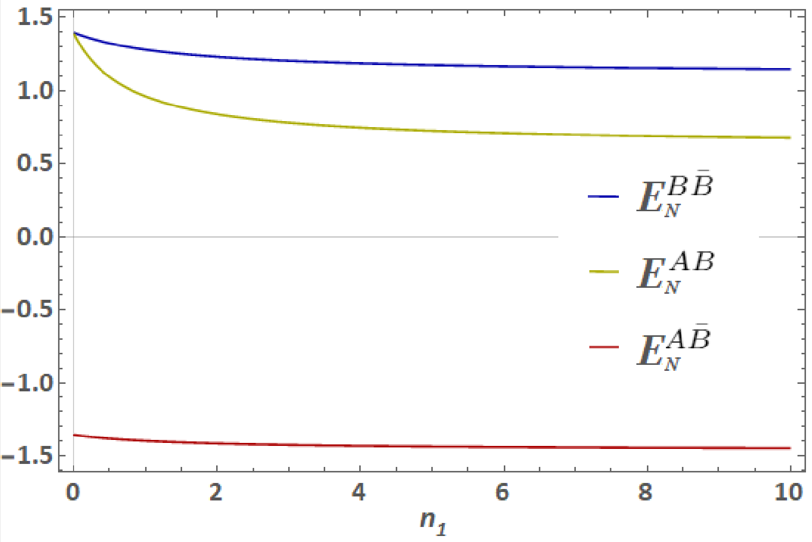

Figure: Gaussian quantum entanglement for different scenarios

Conclusions

We investigated influence of Hawking effect on Gaussian entanglement for massless scalar field modes in presence of a Schwarzschild BH in different scenarios. For a s. consisting of an inertial observer Alice and an accelerated observer Bob, entanglement depends on squeezing parameter of the modes, average thermal photon numbers, Hawking temperature and frequency. Entanglement between Alice and Bob decreases as the Hawking temperature increases. The Hawking radiation induces a thermal noise that causes the decay of quantum entanglement in the system, exhibiting the sudden death behaviour of entanglement. We found out that q. entanglement increases by increasing the squeezing parameter of the modes, tending to a constant value for large values of the squeezing parameter and decreases by increasing the thermal photon numbers. Likewise, in the limit of large frequency of the bosonic field, the Hawking effect tends to vanish, and this means that one can reduce the loss of q. entanglement caused by Hawking effect by increasing the frequency of the bosonic field.

Conclusions

For causally disconnected modes associated with observers Bob and anti-Bob the increase of Hawking temperature strengthens entanglement between the modes and it is also possible even the generation of entanglement due to the effect of Hawking radiation. This different behaviour of entanglement is also contoured by the fact that increase of the squeezing parameter of the modes results in degradation of entanglement. For initial vacuum state ($s = 0$) entanglement is an increasing function of Hawking temperature, meaning that Hawking effect alone is responsible for the generation of entanglement. In other words Gaussian amplification operation gives rise to entanglement as the q. correlations are distributed between accessible and inaccessible information. This interesting result that gives us more information about BHs by measuring the Hawking radiation in a relativistic q. s. Entanglement decreases with the increase of average thermal photon numbers of modes. There is no entanglement between mode of inertial observer Alice and mode associated with imaginary observer anti-Bob.

Thank You!